

# ADDITIVE INVARIANTS FOR KNOTS, LINKS AND GRAPHS IN 3-MANIFOLDS

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**ABSTRACT.** We define two new families of invariants for (3-manifold, graph) pairs which detect the unknot and are additive under connected sum of pairs and  $(-1/2)$  additive under trivalent vertex sum of pairs. The first of these families is closely related to both bridge number and tunnel number. The second of these families is a variation and generalization of Gabai's width for knots in the 3-sphere. We give applications to the tunnel number and higher genus bridge number of connected sums of knots.

## 1. INTRODUCTION

Two of the most basic questions concerning any knot invariant are: “Does it detect the unknot?” and “Is it additive under connected sum?” Among the classical topologically-defined invariants, Seifert genus and bridge number are both well-known for their “yes” answers to both questions. Other invariants such as tunnel number and Gabai width, although they both detect the unknot, have more complicated stories when it comes to additivity. In this paper, we define, for almost any graph in almost any 3-manifold, two new families of invariants which both detect the unknot in  $S^3$  and are additive under connected sum. For graphs, they also satisfy a certain type of additivity under trivalent vertex sum. In this introduction, we give a brief overview of the definition of the invariants (leaving the technicalities until later in the paper), state our main results, and discuss the connection between our invariants and the classical invariants of bridge number, tunnel number, and Gabai width. This work relies on our previous paper [20]. A brief overview of all results we will be needing is provided in Section 2.

**1.1. Notation.** For our purposes, a (3-manifold, graph) pair  $(M, T)$  consists of a compact orientable 3-manifold  $M$  (possibly with boundary) and a properly embedded graph  $T \subset M$ . We make the following:

*Running Assumption:* Every sphere in  $M$  separates, and  $T$  has no vertices of valence 2.

If  $T$  is a 1-manifold properly embedded in a 3-manifold  $M$ , we will write  $M \setminus T$  to mean  $M$  with an open regular neighborhood of  $T$  removed. If  $S \subset M$  is a properly embedded surface, we write  $S \subset (M, T)$  to mean that  $S$  is transverse to  $T$  and  $(M, T) \setminus S$  to indicate the result of removing an open regular neighborhood of  $S$  from both  $M$  and  $T$ . All surfaces appearing in this paper are orientable. A surface  $S \subset (M, T)$  is **essential** if it is incompressible, not  $\partial$ -parallel in the exterior of  $T$ , and not a 2-sphere disjoint from  $T$  bounding a 3-ball disjoint from  $T$ . If  $X$  is a (2 or 3-dimensional) manifold, we use the notation  $Y \sqsubset X$  to mean that  $Y$  is a connected component of  $X$ . Similarly,  $(C, T_C) \sqsubset (M, T) \setminus S$  will indicate that  $C$  is a component of  $M \setminus S$  and  $T_C = T \cap C$ . We say that  $T$  is **irreducible**, if there is no sphere in  $M$  intersecting  $T$  exactly once. We say that  $(M, T)$  is **irreducible** if  $T$  is irreducible and if every (tame) sphere in  $M \setminus T$  bounds a 3-ball in  $M \setminus T$ . For a 3-manifold  $M$ , we let  $g(M)$  denote its Heegaard genus, i.e. the minimal genus of a Heegaard surface for  $M$ . The **tunnel number** for a pair  $(M, T)$  is one less than the Heegaard

genus for the exterior of  $T$ , i.e.  $t(M, T) = g(M \setminus T) - 1$ . A **lens space** is any compact, connected, orientable 3-manifold  $M$  such that  $M \neq S^1 \times S^2$  and  $g(M) = 1$ .

**1.2. The invariants.** As described in [20], a *multiple v.p.-bridge surface* is a closed orientable surface  $\mathcal{H} \subset (M, T)$  such that  $(M, T) \setminus \mathcal{H}$  is the union of simple-to-understand pieces called *v.p.-compressionbodies*. These multiple v.p.-bridge surfaces are generalizations of bridge surfaces for knots in 3-manifolds, Heegaard surfaces for knot exteriors, and the surfaces arising in Scharlemann-Thompson generalized Heegaard splittings of 3-manifolds. The initials “v.p.” stand for “vertex-punctured” and indicate that vertices in  $T$  are treated in a similar way to boundary components of  $M$ .

The components of  $\mathcal{H}$  can be partitioned into two sets: the thick surfaces and the thin surfaces. The union of the thick surfaces is denoted  $\mathcal{H}^+$  and the union of the thin surfaces is denoted  $\mathcal{H}^-$  (so  $H \sqsubset \mathcal{H}^+$  means that  $H$  is a thick surface and  $F \sqsubset \mathcal{H}^-$  means that  $F$  is a thin surface.) We will consider multiple v.p.-bridge surfaces  $\mathcal{H}$  which are *reduced*, *linear*, and *oriented*. Roughly speaking,  $\mathcal{H}$  is oriented if the components of  $\mathcal{H}$  are given coherent transverse orientations which interact nicely with a certain discrete height function;  $\mathcal{H}$  is linear if each component of  $\mathcal{H}^+$  is separating in  $M$ ; and  $\mathcal{H}$  is reduced if there is no “obvious” way of simplifying it. See Section 2 for precise definitions. We let  $\mathbb{H}(M, T)$  denote the set of reduced, linear, oriented multiple v.p.-bridge surfaces for  $(M, T)$ .

If  $(M, T)$  is a (3-manifold, graph)-pair and if  $S \subset (M, T)$  is a surface, we define the **extent** of  $S$  to be

$$\text{ext}(S) = \frac{|S \cap T| - \chi(S)}{2}.$$

If  $S$  is connected, this is simply  $g(S) - 1 + |S \cap T|/2$  where  $g(S)$  is the genus of  $S$ . Two cases are of particular interest: if  $S$  is a minimal bridge sphere for a link  $T \subset S^3$ , then  $\text{ext}(S)$  is one less than the bridge number  $b(T)$  of  $T$ , and if  $S$  is a minimal genus Heegaard surface for the exterior of a link  $T \subset S^3$ , then  $\text{ext}(S)$  is the tunnel number  $t(T)$  of  $T$ . In both cases,  $S$  will meet the requirements for being a v.p.-bridge surface for  $(S^3, T)$ .

For a given  $\mathcal{H} \in \mathbb{H}(M, T)$ , we define the **net extent**  $\text{netext}(\mathcal{H})$  and **width**  $w(\mathcal{H})$  as follows:

$$\begin{aligned} \text{netext}(\mathcal{H}) &= \text{ext}(\mathcal{H}^+) - \text{ext}(\mathcal{H}^-), \\ w(\mathcal{H}) &= 2 \left( \sum_{H \sqsubset \mathcal{H}^+} \text{ext}^2(H) - \sum_{F \sqsubset \mathcal{H}^-} \text{ext}^2(F) \right). \end{aligned}$$

To each multiple v.p.-bridge surface, we can also associate a number, the **net euler characteristic** of  $\mathcal{H}$  which is simply

$$\text{net } \chi(\mathcal{H}) = -\chi(\mathcal{H}^+) + \chi(\mathcal{H}^-).$$

Our two families of invariants are then defined as

$$\begin{aligned} \text{netext}_x(M, T) &= \min_{\mathcal{H}} \text{netext}(\mathcal{H}) \\ w_x(M, T) &= \min_{\mathcal{H}} w(\mathcal{H}). \end{aligned}$$

In both cases, the minimum is taken over all  $\mathcal{H} \in \mathbb{H}(M, T)$  having the property that  $\text{net } \chi(\mathcal{H}) \leq x \leq \infty$ . As noted above for a knot  $K \subset S^3$ ,  $\text{netext}_x(S^3, K)$  is related to classical invariants: for any  $x \geq -2$ , the quantity  $\text{netext}_x(S^3, K)$  is at most  $b(K) - 1$  where  $b(K)$  is the bridge number of  $K$  and, for large enough  $x$ ,  $\text{netext}_x(S^3, K)$  is also at most the tunnel number  $t(K)$  of  $K$ .

The formula for width, on the other hand, is motivated by the well-known formula for Gabai’s width invariant [4] for knots in  $S^3$ . Indeed, Gabai width for  $K \subset S^3$  can be defined as follows.

Consider multiple v.p.-bridge surfaces  $\mathcal{H}$  for  $(S^3, K)$  with the property that the components of  $\mathcal{H}$  are concentric spheres. Then Gabai width is defined as the minimum over all such  $\mathcal{H}$  of the quantity

$$\frac{1}{2} \left( \sum_{H \in \mathcal{H}^+} |H \cap K|^2 - \sum_{F \in \mathcal{H}^-} |F \cap K|^2 \right).$$

Our invariant  $w_{-2}$  for knots in  $S^3$  can be seen as a variant of Gabai width, where we generalize the types of surfaces  $\mathcal{H}$  admitted into the sum and adjust the formula to take into account the euler characteristics of the spheres.

We prove (Corollary 4.5) that, as long as  $x \geq 2g(M) - 2$ , where  $g(M)$  is the Heegaard genus of  $M$ , both  $\text{netext}_x(M, T)$  and  $w_x(M, T)$  are non-negative. Indeed, Theorem 4.10 shows that if  $T$  is a knot and if  $M$  does not have a lens space summand, then  $\text{netext}_x(M, T) = 0$  implies that  $M = S^3$  and  $T$  is the unknot. A similar result holds for width, although we need to add more hypotheses on  $M$  or  $x$ .

Finally, we make a passing comment on the role of  $x$ .

**Remark 1.1.** When working with bridge surfaces or Heegaard surfaces, it is often useful to maintain some control over the euler characteristic of the surfaces. Introducing the parameter  $x$  allows us to do that. Observe that, by the definition, both net extent and width are non-increasing as  $x$  increases. That is, for all  $x \in \mathbb{Z}$ ,

$$\begin{aligned} \text{netext}_x(M, T) &\geq \text{netext}_{x+1}(M, T) \\ w_x(M, T) &\geq w_{x+1}(M, T). \end{aligned}$$

It is easily seen that the values of both  $\text{netext}_x$  and  $w_x$  are integers or half-integers. Thus, the sequences  $(\text{netext}_x(M, T))_x$  and  $(w_x(M, T))_x$  are eventually constant at  $\text{netext}_\infty(M, T)$  and  $w_\infty(M, T)$ .

**1.3. Additivity.** Suppose that  $(\widehat{M}_1, \widehat{T}_1)$  and  $(\widehat{M}_2, \widehat{T}_2)$  are disjoint (3-manifold, graph) pairs such that  $p_1 \in \widehat{T}_1$  and  $p_2 \in \widehat{T}_2$  are either both disjoint from the vertices of  $\widehat{T}_1$  and  $\widehat{T}_2$  or both are trivalent vertices of  $\widehat{T}_1$  and  $\widehat{T}_2$ . Let  $k = 2$  if both are disjoint from the vertices and let  $k = 3$  if both are trivalent vertices. We can form a new (3-manifold, graph) pair

$$(M, T) = (\widehat{M}_1, \widehat{T}_1) \#_k (\widehat{M}_2, \widehat{T}_2)$$

as follows: Remove an open regular neighborhood of  $p_1$  and  $p_2$  from  $(\widehat{M}_1, \widehat{T}_1)$  and  $(\widehat{M}_2, \widehat{T}_2)$  to produce spheres  $P_1$  and  $P_2$  in the boundaries of the resulting pairs  $(M_1, T_1)$  and  $(M_2, T_2)$  respectively. The spheres  $P_1$  and  $P_2$  are both either twice-punctured or thrice-punctured by  $T_1$  and  $T_2$ . Let  $(M, T)$  be the result of gluing the (3-manifold, graph)-pairs together by a homeomorphism  $P_1 \rightarrow P_2$  taking  $T_1 \cap P_1$  to  $T_2 \cap P_2$ . We call the image of  $P_1$  (and  $P_2$ ) in  $(M, T)$  the **summing sphere**. If  $k = 2$ , we say that  $(M, T)$  is the **connected sum** of  $(\widehat{M}_1, \widehat{T}_1)$  and  $(\widehat{M}_2, \widehat{T}_2)$ ; if  $k = 3$ , then  $(M, T)$  is the **trivalent vertex sum** of  $(\widehat{M}_1, \widehat{T}_1)$  and  $(\widehat{M}_2, \widehat{T}_2)$ . We will usually write  $(M, T) = (\widehat{M}_1, \widehat{T}_1) \# (\widehat{M}_2, \widehat{T}_2)$  in place of  $(M, T) = (\widehat{M}_1, \widehat{T}_1) \#_2 (\widehat{M}_2, \widehat{T}_2)$ .

For our purposes, we will say that a pair  $(M, T)$  is **trivial** if it is  $(S^3, T)$  where  $T$  is either an unknot or a trivial  $\theta$ -graph (i.e. a graph having exactly two vertices and exactly three edges each joining the two vertices which can be isotoped into a Heegaard sphere for  $S^3$ .) If  $(M, T) = (\widehat{M}_1, \widehat{T}_1) \#_k (\widehat{M}_2, \widehat{T}_2)$ , then the summing sphere is essential in  $(M, T)$  if neither  $(\widehat{M}_1, \widehat{T}_1)$  or  $(\widehat{M}_2, \widehat{T}_2)$  is trivial.

We say that  $(\widehat{M}_1, \widehat{T}_1), \dots, (\widehat{M}_n, \widehat{T}_n)$  is a **prime decomposition** of  $(M, T)$  if all of the following hold:

- either  $n = 1$  and  $(\widehat{M}_1, \widehat{T}_1) = (M, T)$  or  $n \geq 2$ , and  $(M, T)$  is the result of sequentially connect summing and trivalent vertex summing the  $(\widehat{M}_i, \widehat{T}_i)$  together (in some order).
- For all  $i$ , if  $(\widehat{M}_i, \widehat{T}_i)$  is trivial, then  $(\widehat{M}_i, \widehat{T}_i) = (S^3, \text{trivial } \theta\text{-graph})$  and only connected sums are performed on  $(\widehat{M}_i, \widehat{T}_i)$ .
- For all  $i$ , if  $P \subset (\widehat{M}_i, \widehat{T}_i)$  is an essential sphere, then either  $P \cap \widehat{T}_i = \emptyset$  or  $|P \cap \widehat{T}_i| \geq 4$ .

Since we require that the summing be done sequentially, the graph in  $M$  dual to the summing spheres is a tree. Under the assumption that no sphere in  $M$  is non-separating, that  $T$  is a knot, and that  $(M, T)$  is non-trivial, then Miyazaki [9, Theorem 4.1] has shown that  $(M, T)$  has a unique prime factorization, up to re-ordering. This was extended to the situation where  $T$  is a  $\theta$ -graph by Matveev and Turaev [8].

Let  $\mathbb{M}$  be the set whose elements are irreducible (3-manifold, graph) pairs  $(M, T)$  with  $M$  connected,  $T$  non-empty, and with the property that  $T$  intersects each sphere in  $\partial M$  at least three times. Let  $\mathbb{M}_2 \subset \mathbb{M}$  be the subset where  $g(M) \leq 2$  and let  $\mathbb{M}_s \subset \mathbb{M}$  be the subset where every closed surface in  $M$  separates. We prove:

**Theorem 5.7** (Additivity Theorem). *Let  $(M, T) \in \mathbb{M}$  be non-trivial, and let  $x$  be any integer with  $x \geq 2g(M) - 2$ . Then there is a prime factorization of  $(M, T)$  into  $(\widehat{M}_1, \widehat{T}_1), \dots, (\widehat{M}_n, \widehat{T}_n)$  so that there exist integers  $x_1, \dots, x_n$ , summing to at most  $x - 2(n - 1)$ , and*

$$\text{netext}_x(M, T) = -p_3/2 + \sum_{i=1}^n \text{netext}_{x_i}(\widehat{M}_i, \widehat{T}_i).$$

where  $p_3$  is the number of thrice punctured spheres in the decomposition. Furthermore, if  $(M, T) \in \mathbb{M}_s$  or if  $(M, T) \in \mathbb{M}_2$  and  $x \leq 2$ , then also

$$w_x(M, T) = -p_3/2 + \sum_{i=1}^n w_{x_i}(\widehat{M}_i, \widehat{T}_i).$$

The result for width is particularly striking. For many years, it was an open question as to whether or not Gabai width satisfied an additivity property with respect to connected sum of knots. Indeed, for some classes of knots [12], Gabai width is additive. In [16], however, Scharlemann and Thompson proposed counter-examples to the additivity of Gabai width and in [1], Blair and Tomova proved that one of those proposed counter-examples is actually a counter-example. On the other hand, Theorem 5.7 shows that our invariant  $w_{-2}$ , which is a slightly modified version of Gabai width, is additive under connected sum and, even more surprisingly, “higher genus” widths are also additive. For more details, see Section 6.

**1.4. Applications to classical invariants.** We give several simple applications of our results to knots in 3-manifolds. For the statement, recall that a knot  $K$  in a 3-manifold  $M$  is **meridionally small** or **m-small** if there is no surface  $S \subset (M, K)$  such that  $S \cap K \neq \emptyset$  and  $S$  is essential in  $(M, K)$  (i.e. is incompressible and not  $\partial$ -parallel in the exterior of  $K$ .)

We give two short proofs (Theorem 7.1 and Theorem 7.2) of classical results of Schubert [17] and Norwood [11] showing that 2-bridge knots and tunnel number 1 knots (more generally) are both prime. Scharlemann and Schultens [14] generalized Norwood’s result to show that the tunnel number of the connected sum of  $n$  knots is at least  $n$ . Morimoto [10] proved a stronger result for m-small knots: the tunnel number of the connected sum of  $n$  m-small knots is at least the sum of the tunnel numbers of the factors. We prove a theorem which combines the Scharlemann-Schultens and Morimoto results. Dropping the hats off the summands for convenience, the statement is:

**Theorem 7.3.** *For each  $i \in \{1, \dots, n\}$  let  $K_i$  be a knot in a closed, orientable 3-manifold  $M_i$  such that every sphere in  $M_i$  separates and each  $(M_i, K_i)$  is prime. Assume that there is an integer  $j \leq n$  so that  $K_i$  is  $m$ -small if and only if  $i \leq j$ . Then, letting  $(M, K) = (M_1, K_1) \# \dots \# (M_n, K_n)$ , we have:*

$$(n - j) + t(K_1) + \dots + t(K_j) \leq t(K) \leq (n - 1) + \sum t(K_i)$$

Kobayashi-Rieck [6] studied the asymptotic properties of tunnel number of the connected sum  $nK$  of a knot  $K$  with itself  $n$  times. As part of that project, they showed that for “admissible”  $m$ -small knots  $K$ ,

$$0 \leq \lim_{n \rightarrow \infty} \frac{t(nK) - nt(K)}{n - 1} < 1.$$

Along the way, they prove the left-hand inequality holds for each term of the sequence (not just in the limit.) Our Theorem 7.3, gives another proof that  $0 \leq \frac{t(nK) - nt(K)}{n - 1}$  for  $m$ -small knots.

If  $K \subset M$  is a link, a surface  $S$  is a **genus  $g$  bridge surface** for  $K$  if after removing a regular neighborhood of all components of  $K$  that are disjoint from  $S$ ,  $S$  is a Heegaard surface for the resulting manifold which intersects  $K$  transversally and divides  $K$  into arcs parallel into  $S$ . The **genus  $g$  bridge number** of  $(M, K)$  is the smallest natural number  $b_g(K)$  such that there is a genus  $g$  bridge surface for  $K$ .

**Remark 1.2.** This is not quite the definition given by Doll [3] for higher genus bridge number. He defines  $b_g(K)$  only when it is positive. Subsequently, opinions have differed on how to extend the definition to allow  $b_g(K) = 0$ . Some authors (as we do) declare  $b_g(K) = 0$  if and only if  $K$  is a core loop for a genus  $g$  Heegaard splitting and others if and only if  $K$  is isotopic into a genus  $g$  Heegaard surface for  $M$ .

A knot  $K$  in a 3-manifold  $M$  is **small** if  $M \setminus K$  contains no closed essential surfaces. By [2, Theorem 2.0.3], a small knot in  $S^3$  is also  $m$ -small, but we will not use that fact. We show that this higher genus bridge number satisfies a certain super-additivity for small knots, in the following sense:

**Theorem 1.3.** *Suppose that  $(M_i, K_i)$  are small and  $m$ -small for  $i \in \{1, \dots, n\}$ . Let  $(M, K) = \#_{i=1}^n (M_i, K_i)$  and let  $g \geq g(M, K)$ . Then there exist  $g_i$  such that  $\sum g_i \leq g$ ,  $g_i \geq g(M_i, K_i)$  and*

$$\sum_{i=1}^n (g_i + b_{g_i}(K) - 1) \leq g + b_g(K) - 1.$$

Restricting to two summands for convenience, we have:

**Theorem 1.4.** *Suppose that for  $i = 1, 2$  the pair  $(M_i, K_i)$  is small and  $m$ -small. Let  $g_1 = g(M_1)$  and  $g_2 = g(M_2)$  and assume that  $t(K_i) \geq g_i$  for  $i \in \{1, 2\}$ . Let  $(M, K) = (M_1, K_1) \# (M_2, K_2)$  and let  $g = g(M)$ . Then:*

$$b_g(M, K) = b_{g_1}(M_1, K_1) + b_{g_2}(M_2, K_2) - 1.$$

*Proof.* By Theorem 1.3, given  $g \geq g(M_1 \# M_2)$ , there exist  $g_1$  and  $g_2$  such that  $g_1 + g_2 \leq g$  and so that

$$(g_1 + b_{g_1}(K_1) - 1) + (g_2 + b_{g_2}(K_2) - 1) \leq g + b_g(K) - 1.$$

Since  $g(M_1) + g(M_2) = g$ , we actually have  $g_1 + g_2 = g$ . Thus,

$$(b_{g_1}(K_1) - 1) + (b_{g_2}(K_2) - 1) \leq b_g(K) - 1.$$

By adapting Lemma [3, Bridge Inequality 1.2] to our definitions, we have

$$b_g(K) \leq \max(b_{g_1}(K_1), 1) + \max(b_{g_2}(K_2), 1) - 1.$$

Since  $t(K_i) \geq g_i$ , we have:

$$b_g(K) \leq b_{g_1}(K_1) + b_{g_2}(K_2) - 1.$$

Thus,

$$b_g(K) = b_{g_1}(K_1) + b_{g_2}(K_2) - 1$$

□

This solves Doll's Conjecture 1.1 for small knots  $K_1$  and  $K_2$  and for  $g = g(M_1) + g(M_2)$ .

## 2. PRELIMINARIES

We begin by reviewing important definitions and results from [20] which are essential to the discussion in this paper. We will be interested in a generalization of the usual definition of a compressing disc. Namely:

**Definition 2.1.** Suppose that  $S \subset (M, T)$  is a surface and that  $D$  is an embedded disc in  $M$  such that the following hold:

- (1)  $\partial D \subset (S \setminus T)$ , the interior of  $D$  is disjoint from  $S$ , and  $D$  is transverse to  $T$ .
- (2)  $|D \cap T| \leq 1$ .
- (3) There is no disc  $E \subset S$  such that  $\partial E = \partial D$  and  $E \cup D$  bounds either a 3-ball in  $M$  disjoint from  $T$  or a 3-ball in  $M$  whose intersection with  $T$  consists entirely of a single unknotted arc with one endpoint in  $E$  and one endpoint in  $D$ .

Then  $D$  is an **sc-disc**. If  $|D \cap T| = 0$  and  $\partial D$  does not bound a disc in  $S \setminus T$ , then  $D$  is a **compressing disc**. If  $|D \cap T| = 0$  and  $\partial D$  does bound a disc in  $S \setminus T$ , then  $D$  is a **semi-compressing disc**. If  $|D \cap T| = 1$  and  $\partial D$  does not bound an unpunctured disc or a once-punctured disc in  $S \setminus T$ , then  $D$  is a **cut disc**. If  $|D \cap T| = 1$  and  $\partial D$  does bound an unpunctured disc or a once-punctured disc in  $S \setminus T$ , then  $D$  is a **semi-cut disc**. A **c-disc** is a compressing disc or cut disc. The surface  $S \subset (M, T)$  is **c-incompressible** if  $S$  does not have a c-disc; it is **c-essential** if it is essential and c-incompressible. If  $S$  is separating and there is a pair of disjoint sc-discs on opposite sides of  $S$ , then  $S$  is **sc-weakly reducible**, otherwise it is **sc-strongly irreducible**.

As we explained in the introduction, we will be interested in certain surfaces  $\mathcal{H}$  which cut  $(M, T)$  into simple-to-understand pieces. These pieces are called v.p.-compressionbodies.

**Definition 2.2.** Suppose that  $F$  is a closed, connected, orientable surface. We say that  $(F \times I, T)$  is a **trivial product compressionbody** if  $T$  is isotopic to the union of vertical arcs. We let  $\partial_{\pm}(F \times I) = F \times \{\pm 1\}$ . If  $B$  is a 3-ball and if  $T \subset B$  is a (possibly empty) connected, properly embedded,  $\partial$ -parallel tree, having at most one interior vertex, then we say that  $(B, T)$  is a **trivial ball compressionbody**. We let  $\partial_+ B = \partial B$  and  $\partial_- B = \emptyset$ . A **trivial compressionbody** is either a trivial product compressionbody or a trivial ball compressionbody.

A pair  $(C, T)$ , with  $C$  connected, is a **v.p.-compressionbody** if there is some component denoted  $\partial_+ C$  of  $\partial C$  and a collection of pairwise disjoint sc-discs  $\mathcal{D} \subset (C, T)$  for  $\partial_+ C$  such that the result of  $\partial$ -reducing  $(C, T)$  using  $\mathcal{D}$  is a union of trivial compressionbodies. The set of sc-discs  $\mathcal{D}$  is called a **complete collection of sc-discs for  $(C, T)$** . The set  $\partial C \setminus \partial_+ C$  is denoted by  $\partial_- C$ .

An edge of  $T$  which is disjoint from  $\partial_+ C$  is a **ghost arc**. An edge of  $T$  with one endpoint in  $\partial_+ C$  and one in  $\partial_- C$  is a **vertical arc**. A component of  $T$  which is an arc having both endpoints on  $\partial_+ C$  is a **bridge arc**. A component of  $T$  which is homeomorphic to a circle and is disjoint from  $\partial C$  is called a **core loop**. A **bridge disc** for  $\partial_+ C$  in  $C$  is an embedded disc in  $C$  with boundary



the union of two arcs  $\alpha$  and  $\beta$  such that  $\alpha \subset \partial_+ C$  joins distinct points of  $\partial_+ C \cap T$  and  $\beta$  is the union of edges of  $T$ . We will only consider bridge discs which are disjoint from the vertices of  $T$ .

Figure 1 depicts a v.p.-compressionbody  $(C, T)$  containing three vertical arcs, one ghost arc, one bridge arc, and one core loop. If  $(C, T)$  is a v.p.-compressionbody such that  $T$  has no interior vertices, then every component of  $T$  is either a vertical arc, ghost arc, bridge arc, or core loop. We will often reduce to this situation by drilling out vertices of  $T$  (i.e. removing a regular neighborhood of them so that vertices correspond to new spherical boundary components of the resulting 3-manifold.)

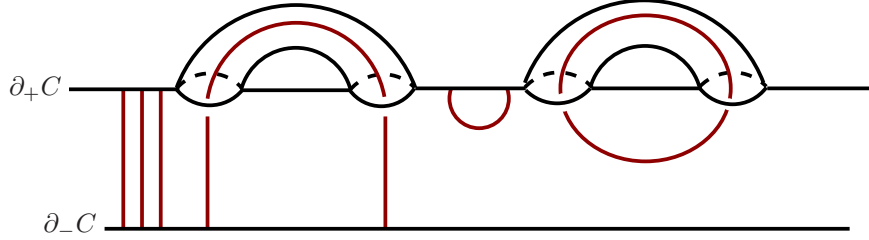


FIGURE 1. A v.p.-compressionbody  $(C, T)$ . From left to right we have three vertical arcs, one ghost arc, one bridge arc, and one core loop in  $T$ . This figure was reappropriated from [20].

What follows is a key property of v.p.-compressionbodies which we will use on several occasions.

**Lemma 2.3** ([20, Lemma 3.4]). *Suppose that  $(C, T)$  is a v.p.-compressionbody such that no component of  $\partial_- C$  is a 2-sphere intersecting  $T$  exactly once. The following are true:*

- (1)  $(C, T)$  is a trivial compressionbody if and only if there are no sc-discs for  $\partial_+ C$ .
- (2) If  $D$  is an sc-disc for  $\partial_+ C$ , then reducing  $(C, T)$  using  $D$  is the union of v.p.-compressionbodies. Furthermore, there is a complete collection of sc-discs for  $(C, T)$  containing  $D$ .

**Definition 2.4.** A **multiple v.p.-bridge surface** for  $(M, T)$  is a closed (possibly disconnected) surface  $\mathcal{H} \subset (M, T)$  such that:

- $\mathcal{H}$  is the disjoint union of  $\mathcal{H}^-$  and  $\mathcal{H}^+$ , each of which is the union of components of  $\mathcal{H}$ ;
- $(M, T) \setminus \mathcal{H}$  is the union of embedded v.p.-compressionbodies  $(C_i, T_i)$  with  $\mathcal{H}^- \cup \partial M = \bigcup \partial_- C_i$  and  $\mathcal{H}^+ = \bigcup \partial_+ C_i$ ;
- Each component of  $\mathcal{H}$  is adjacent to two distinct v.p.-compressionbodies.

The components of  $\mathcal{H}^-$  are called **thin surfaces** and the components of  $\mathcal{H}^+$  are called **thick surfaces**. If  $\mathcal{H}$  is connected, then  $\mathcal{H} = \mathcal{H}^+$  is called a **v.p.-bridge surface** for  $(M, T)$ .

Observe that, for a multiple v.p.-bridge surface  $\mathcal{H}$  of  $(M, T)$ , each component of  $\mathcal{H}^+$  is a v.p.-bridge surface for the component of  $(M, T) \setminus \mathcal{H}^-$  containing it.

We are usually interested in multiple v.p.-bridge surfaces that have certain additional properties:

**Definition 2.5** (for details, see [20, Section 3.2]). Suppose that  $\mathcal{H}$  is a multiple v.p.-bridge surface for  $(M, T)$ . Suppose that each component of  $\mathcal{H}$  is given a transverse orientation so all orientations are consistent on the boundary of each v.p.-compressionbody. Furthermore, suppose there is a function  $f: \mathcal{H} \rightarrow \mathbb{N}$ , constant on each component of  $\mathcal{H}$  so whenever there is an oriented path in  $M$

from  $S_1 \in \mathcal{H}$  to  $S_2 \in \mathcal{H}$  that is transverse to  $\mathcal{H}$  and oriented consistently with every component of  $\mathcal{H}$  it intersects then  $f(S_1) < f(S_2)$ . If such a function exists  $\mathcal{H}$  is **oriented** and  $f$  is called a **height function** for  $\mathcal{H}$ . We say that  $\mathcal{H}$  is **linear** if each component of  $\mathcal{H}^+$  separates  $M$ .

Just as a Heegaard surface for a 3-manifold can be stabilized and so have higher genus than necessary, so a multiple v.p.-bridge surface may have thick surfaces that are higher genus or have more punctures than necessary. In [20], we defined a collection of destabilizing moves for multiple v.p.-bridge surfaces. The types of destabilization for  $H \sqsubset \mathcal{H}^+$  are as follows (see [20] for precise definitions). All of these are called **generalized stabilizations**.

- (Destabilization) Compressing along a certain compressing disc for  $H$  having boundary which is non-separating on  $H$ .
- (Meridional Destabilization) Compressing along a certain cut disc for  $H$  having boundary which is non-separating on  $H$ .
- (Boundary Destabilization) Compressing along a certain separating compressing disc for  $H$  and discarding a component of the resulting surface.
- (Meridional Boundary Destabilization) Compressing along a certain separating cut disc for  $H$  and discarding a component of the resulting surface.
- (Ghost Boundary Destabilization) Compressing along a certain separating compressing disc for  $H$  and discarding a component of the resulting surface.
- (Ghost Meridional Boundary Destabilization) Compressing along a certain separating cut disc for  $H$  and discarding a component of the resulting surface.

The operations of destabilization and boundary-destabilization are versions of well-known operations on Heegaard splittings. Meridional destabilization and meridional boundary destabilization are essentially the same, except that a cut disc plays the role of the compressing disc. A ghost (meridional) boundary destabilization is the same as (meridional) boundary destabilization after removing an open regular neighborhood of a certain subgraph of  $T$  from  $(M, T)$ .

There are times when it is possible to isotope a component of  $\mathcal{H}^+$  across a bridge disc so as to reduce the number of intersections between  $\mathcal{H}$  and  $T$  while still producing a multiple v.p.-bridge surface. Two are of special interest (see Figure 2):

- (Unperturbing) Isotope  $H \sqsubset \mathcal{H}^+$  across a bridge disc  $D$  which shares a single point of intersection with a bridge disc on the opposite side of  $H$ . The result of this isotopy is that the number of intersections of  $H$  and  $T$  is reduced by two.
- (Undoing a removable arc) Isotope  $H \sqsubset \mathcal{H}^+$  across a bridge disc  $D$  that has a single point of intersection with a complete set of sc-discs on the other side of  $H$  such that the point of intersection lies on a compressing or semi-compressing disc. The result of this isotopy is that the number of intersections of  $H$  and  $T$  is reduced by two.

In [20, Section 4], we show that the result of performing any of the generalized destabilizations, unperturbing, or removing a removable arc is still a (linear, oriented) multiple v.p.-bridge surface.

There are two other ways of “simplifying” a multiple v.p.-bridge surface. In some ways, these play the most important role in the theory.

**Definition 2.6.** Suppose that  $\mathcal{H}$  is a multiple v.p.-bridge surface for  $(M, T)$  and that some  $(P, T_P) \sqsubset (M, T) \setminus \mathcal{H}$  is a trivial product compressionbody adjacent to  $\mathcal{H}^-$  (rather than  $\partial M$ ). Then  $\mathcal{H}' = \mathcal{H} \setminus \partial P$  is obtained from  $\mathcal{H}$  by a **consolidation** (or by **consolidating**  $\mathcal{H}$ .)



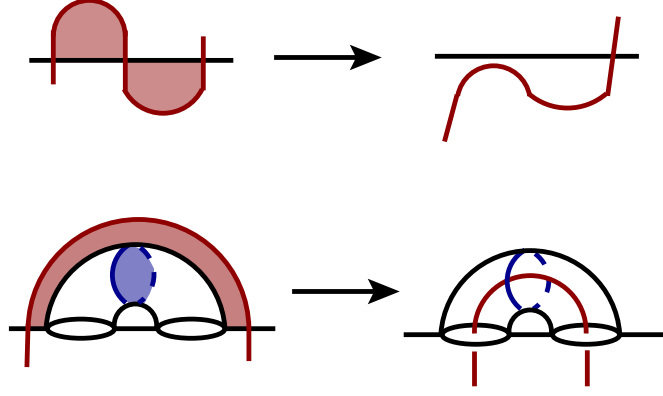


FIGURE 2. The top row gives a schematic depiction of unperturbing. The bottom row gives a schematic depiction of removing a removable arc.

Lemma 5.6 of [20] shows that if  $\mathcal{H}'$  is obtained by consolidating  $\mathcal{H}$ , then it is still a multiple v.p.-bridge surface. If  $\mathcal{H}$  was oriented or linear then so is  $\mathcal{H}'$ . We say that a multiple v.p.-bridge surface  $\mathcal{H}$  is **reduced** if it is impossible to perform a generalized destabilization or consolidation and if it is impossible to unperturb it or undo a removable arc. Let  $\mathbb{H}(M, T)$  denote all reduced, oriented, linear multiple v.p.-bridge surfaces, with two surfaces being equivalent if they are isotopic via an isotopy transverse to  $T$ .

**Definition 2.7.** Suppose that  $\mathcal{H}$  is an oriented v.p.-bridge surface such that  $H \sqsubset \mathcal{H}^+$  is sc-weakly reducible in  $(M, T) \setminus \mathcal{H}^-$ . Let  $D_-$  and  $D_+$  be disjoint sc-discs on opposite sides of  $H$ . Let  $H_\pm$  be the result of compressing  $H$  using  $D_\pm$  and performing a small isotopy to the side of  $H$  containing  $D_\pm$ . Let  $F$  be the result of compressing  $H$  using both  $D_-$  and  $D_+$ . Let  $\mathcal{J}^+ = (\mathcal{H}^+ \setminus H) \cup (H_- \cup H_+)$  and  $\mathcal{J}^- = \mathcal{H}^- \cup F$ . Then  $\mathcal{J} = \mathcal{J}^+ \cup \mathcal{J}^-$  is obtained by **untelescoping**  $\mathcal{H}$ . Figure 3 gives an example.

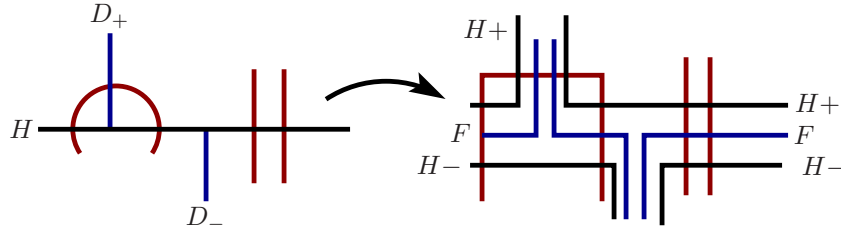


FIGURE 3. Untelescoping  $H$ . The red curves are portions of  $T$ . The blue lines on the left are sc-discs for  $H$ . Note that if a semi-cut or cut disc is used then a ghost arc is created. This figure was reappropriated from [20].

Lemma 5.2 and Definition 5.3 of [20] show that untelescoping an oriented multiple v.p.-bridge surface results in an oriented multiple v.p.-bridge surface. We need the following:

**Lemma 2.8** (Lemma 5.8 of [20]). *Suppose that  $H$  is an oriented multiple v.p.-bridge surface for  $(M, T)$  with  $(H_\downarrow, T_\downarrow)$  and  $(H_\uparrow, T_\uparrow)$  the v.p.-compressionbodies of  $(M, T) \setminus H$  on either side. Suppose that  $D_\uparrow$  and  $D_\downarrow$  are an sc-weak reducing pair. Let  $H_- \subset H_\downarrow$  and  $H_+ \subset H_\uparrow$  be the new thick surfaces created by untelescoping  $H$ . Let  $F$  be the union of the new thin surfaces. Then the following are equivalent for a component  $\Phi$  of  $F$ :*

- (1)  $\Phi$  is adjacent only to a remnant of  $D_\uparrow$  (or  $D_\downarrow$ , respectively).
- (2) The disc  $D_\uparrow$  (or  $D_\downarrow$ , respectively) is separating and  $\Phi$  bounds a trivial product v.p.-compressionbody in  $H_\uparrow$  (or  $H_\downarrow$ , respectively) with a component of  $H_+$  (or  $H_-$ , respectively).

Suppose that  $\mathcal{H} \in \mathbb{H}(M, T)$ . An **elementary thinning sequence** consists of the following operations, in order:

- (1) untelescoping a component of  $\mathcal{H}^+$  to create an oriented multiple v.p.-bridge surface  $\mathcal{H}_1$ ;
- (2) consolidating components of  $\mathcal{H}_1 \setminus \mathcal{H}$  that cobound trivial product compressionbodies in  $(M, T) \setminus \mathcal{H}_1$  to create  $\mathcal{H}_2$
- (3) consolidating components of  $\mathcal{H}_2^+ \setminus \mathcal{H}^+$  and components of  $\mathcal{H}^- \subset \mathcal{H}_2^-$  that cobound trivial product compressionbodies in  $(M, T) \setminus \mathcal{H}_2$  to create  $\mathcal{H}' = \mathcal{H}_3$ .

See [20, Figure 11] for a schematic depiction of an elementary thinning sequence. It follows from [20, Corollary 5.10] that the result of applying an elementary thinning sequence to  $\mathcal{H} \in \mathbb{H}(M, T)$  is a linear oriented multiple v.p.-bridge surface having the property that no consolidation is possible. It may, however, be possible to destabilize, unperturb, or undo a removable arc.

**Definition 2.9** (Definition 7.2 of [20]). Suppose that  $\mathcal{H} \in \mathbb{H}(M, T)$  is reduced and that  $T$  is irreducible. An **extended thinning move** applied to  $\mathcal{H}$  consists of the following steps in the following order:

- (1) Perform an elementary thinning sequence
- (2) Destabilize, unperturb, and undo removable arcs until no generalized stabilizations, perturbations, or removable arcs remain
- (3) Perform as many consolidations as possible
- (4) Repeat (2) and (3) as much as possible.

In [20, Remark 7.3] we explain why Steps (2), (3), and (4) are guaranteed to terminate. It is then evident from the definition that the result of applying an extended thinning move to  $\mathcal{H} \in \mathbb{H}(M, T)$  is also an element of  $\mathbb{H}(M, T)$ . For  $\mathcal{H}, \mathcal{K} \in \mathbb{H}(M, T)$ , we say that  $\mathcal{H}$  **thins to**  $\mathcal{K}$  and write  $\mathcal{H} \rightarrow \mathcal{K}$  if there is a (possibly empty) sequence of extended thinning moves producing  $\mathcal{K}$  from  $\mathcal{H}$ . If  $\mathcal{H} \rightarrow \mathcal{K}$  implies that  $\mathcal{H} = \mathcal{K}$  (equivalently, no extended thinning move can be applied to  $\mathcal{H}$ ) then we say that  $\mathcal{H}$  is **locally thin**. We proved (in the absence of the assumption that every sphere in  $M$  separates):

**Theorem 2.10** (Theorem 7.4 of [20]). *Suppose that  $(M, T)$  is a 3-manifold graph pair such that no component of  $\partial M$  is a sphere intersecting  $T$  exactly two or fewer times. Then  $\rightarrow$  is a partial order on  $\mathbb{H}(M, T)$  and for every  $\mathcal{H} \in \mathbb{H}(M, T)$  there exists a locally thin  $\mathcal{K} \in \mathbb{H}(M, T)$  such that  $\mathcal{H} \rightarrow \mathcal{K}$ .*

Locally thin multiple v.p.-bridge surfaces have some particularly nice properties:

**Theorem 2.11** (Properties of locally thin surfaces). *Suppose that  $(M, T)$  is a (3-manifold, graph) pair with  $T$  irreducible. Let  $\mathcal{K} \in \mathbb{H}(M, T)$  be locally thin. Then the following hold:*

- (1) Each component of  $\mathcal{K}^+$  is sc-strongly irreducible in  $(M, T) \setminus \mathcal{K}^-$ .
- (2) Every component of  $\mathcal{K}^-$  is c-essential in  $(M, T)$ .
- (3) If  $(M, T)$  is irreducible and if  $\mathcal{K}$  contains a 2-sphere disjoint from  $T$ , then  $T = \emptyset$  and  $M = S^3$  or  $M = B^3$ .
- (4) Suppose that  $P \subset (M, T)$  is an essential sphere such that  $|P \cap T| \leq 3$ . Then some  $F \sqsubset \mathcal{K}^-$  is an essential sphere with  $|F \cap T| \leq |P \cap T|$  and  $F$  can be obtained from  $P$  by a sequence of isotopies and compressions using sc-discs.

See [20, Theorem 8.6] and [20, Theorem 9.2] for proofs. As with Theorem 2.10, this theorem also holds in the absence of the “every sphere in  $M$  separates” assumption.

### 3. NET EXTENT, WIDTH, AND THINNING SEQUENCES

**3.1. Effects of thinning.** In this section we show that net extent and width do not increase under thinning of a linear oriented multiple v.p.-bridge surface.

**Lemma 3.1.** *Suppose that  $\mathcal{H}$  is an oriented, multiple v.p.-bridge surface for  $(M, T)$  and that  $\mathcal{K}$  is obtained by an elementary thinning sequence from  $\mathcal{H}$ . Then  $\text{net } \chi(\mathcal{K}) = \text{net } \chi(\mathcal{H})$  and  $\text{netext}(\mathcal{K}) = \text{netext}(\mathcal{H})$ . Furthermore, if the following hold:*

- $T$  is irreducible, and
- either at least one of discs in the weak reducing pair has boundary which separates  $\mathcal{H}^+$  or the union of the boundaries of the discs in the weak-reducing pair is non-separating on  $\mathcal{H}^+$

then  $w(\mathcal{K}) \leq w(\mathcal{H})$ .

*Proof.* The statement for net euler characteristic is similar to that of [14, Lemma 2]; it is easily verified by examining the definition of elementary thinning sequence. We take on the proof that net extent does not change and that width does not increase under an elementary thinning sequence. Observe that consolidation will never change net euler characteristic, net extent, or width.

Let  $H \subset \mathcal{H}^+$  be the thick surface which is untelescoped using a weak reducing pair  $\{D_-, D_+\}$ . Let  $i = |D_+ \cap T|$  and  $j = |D_- \cap T|$  (so  $i, j \in \{0, 1\}$ ). Let  $\mathcal{H}_1$  be the surface obtained by untelescoping. The surface  $\mathcal{K}$  is obtained from  $\mathcal{H}_1$  by consolidations so  $\text{netext}(\mathcal{H}_1) = \text{netext}(\mathcal{K})$  and  $w(\mathcal{H}_1) = w(\mathcal{K})$ . It suffices to show then that  $\text{netext}(\mathcal{H}_1) = \text{netext}(\mathcal{H})$  and  $w(\mathcal{H}_1) \leq w(\mathcal{H})$ .

Let  $H_+$  be the union of the components  $\mathcal{H}_1^+$  resulting from compressing  $H$  using  $D_+$  (there are at most two such components.) Let  $H_-$  be the union of the thick surfaces in  $\mathcal{H}_1^+$  which result from compressing  $H$  using  $D_-$ . Let  $F$  be the union of the new thin surfaces (i.e. the components of  $\mathcal{H}_1^- \setminus \mathcal{H}^-$ .) We have

$$\begin{aligned} \text{ext}(H_+) &= \text{ext}(H) + i - 1 \\ \text{ext}(H_-) &= \text{ext}(H) + j - 1 \\ \text{ext}(F) &= \text{ext}(H) + i + j - 2. \end{aligned}$$

Consequently,

$$\text{ext}(H_+) + \text{ext}(H_-) - \text{ext}(F) = \text{ext}(H),$$

and so  $\text{netext}(\mathcal{H}_1) = \text{netext}(\mathcal{H})$ . Since extent is linear over components, we have  $\text{netext}(\mathcal{H}) = \text{netext}(\mathcal{K})$ .

We need to exert more care with width. Assume, therefore, the two additional hypotheses in the statement of the lemma. The second new hypothesis guarantees that  $F = \mathcal{K}^- \setminus \mathcal{H}^-$  is connected (Lemma 2.8). Let  $H'_+$ ,  $H'_-$ , and  $F'$  be the components of  $H_\pm \cap \mathcal{K}$  and  $F \cap \mathcal{K}$  respectively. Let  $H''_\pm$  be a component of  $H_\pm$  which is consolidated with a component  $F''_\pm$  of  $F$ .

Let  $x = \text{ext}(H)$ . Let  $x'_\pm$  and  $x''_\pm$  be the extents of  $H'_\pm$  and  $H''_\pm$  respectively. Let  $y$  be the extent of  $F'$ . Note that the extents of the components of  $F$  which are consolidated are exactly  $x''_+$  and  $x''_-$ . Then

$$\begin{aligned} x'_+ &= x + i - 1 - x''_+ \\ x'_- &= x + j - 1 - x''_- \\ y &= x + i + j - 2 - x''_+ - x''_- \end{aligned}$$

Algebra then shows that

$$(x'_+)^2 + (x'_-)^2 - y^2 = x^2 - 2((j-1) - x''_-)((i-1) - x''_+).$$

Thus,

$$\frac{1}{2} (w(\mathcal{K}) - w(\mathcal{H})) = -2((j-1) - x''_-)((i-1) - x''_+).$$

This is non-positive, as desired, unless exactly one of  $((j-1) - x''_-)$  or  $((i-1) - x''_+)$  is positive and the other is negative. Without loss of generality, suppose

$$x''_- < j-1 \in \{-1, 0\}.$$

Since  $T$  is irreducible, and since  $H''_-$  is connected, by the definition of extent,  $H''_-$  is a sphere disjoint from  $T$ . Hence,  $x''_- = -1$  and  $j = 0$ . Thus,

$$\frac{1}{2} (w(\mathcal{K}) - w(\mathcal{H})) = -2(-1+1)((i-1) - x''_+) \leq 0,$$

as desired.  $\square$

**Remark 3.2.** The reason for the second additional assumption for the result on width in Lemma 3.1 is due to the fact that if (using the notation from the proof)  $F'$  is disconnected, then the difference  $w(\mathcal{K}) - w(\mathcal{H})$  is given by

$$((x'_+)^2 + (x'_-)^2 - y_1^2 - y_2^2) - x^2$$

where  $y_1$  and  $y_2$  are the extents of the components of  $F'$  instead of  $((x'_+)^2 + (x'_-)^2 - y^2) - x^2$ . That distinction is enough to make the proof not go through in the case when  $F'$  is disconnected.

For convenience, if  $\mathcal{H}$  is a linear multiple v.p.-bridge surface for  $(M, T)$  we define the **width hypothesis** to be both of the following assumptions:

- (W1)  $T$  is irreducible
- (W2) Either each component of  $\mathcal{H}^+$  has genus at most 2 or every closed surface in  $M$  separates

Of course, we continue to employ the “running assumption,” without remarking on it.

**Corollary 3.3.** *Suppose that  $\mathcal{H}, \mathcal{K} \in \mathbb{H}(M, T)$  and that  $\mathcal{H} \rightarrow \mathcal{K}$ . Then  $\text{net } \chi(\mathcal{K}) \leq \text{net } \chi(\mathcal{H})$  and  $\text{netext}(\mathcal{K}) \leq \text{netext}(\mathcal{H})$ . Additionally, if the width hypothesis holds for  $\mathcal{H}$ , then  $w(\mathcal{K}) \leq w(\mathcal{H})$ .*

*Proof.* It is easily verified that consolidation, destabilization of all kinds, as well as unperturbing and eliminating a removable arc do not increase net euler characteristic, net extent, or width. By Lemma 3.1, therefore, a thinning sequence does not increase net extent.

Assume, therefore, that the width hypothesis holds for  $\mathcal{H}$ . Observe that consolidation, elementary thinning sequences, destabilization of all kinds, as well as unperturbing and eliminating a removable arc do not change these properties. We will show that the width hypotheses imply that  $\mathcal{H}$  satisfies the hypothesis in the second bullet point of the statement of Lemma 3.1. It then follows that  $w(\mathcal{K}) \leq w(\mathcal{H})$ .

Suppose that  $D_-$  and  $D_+$  are a weak reducing pair for  $H \sqsubset \mathcal{H}^+$ . If one of  $D_-$  or  $D_+$  has boundary which separates  $H$ , then we are done. Assume, therefore, that both  $\partial D_-$  and  $\partial D_+$  are non-separating on  $H$ . This implies  $H$  is not a sphere. By (W2), either  $g(H) \leq 2$  or every closed surface in  $M$  separates.

Assume, first, that  $g(H) \leq 2$ . We already know that  $g(H) \neq 0$ . If  $g(H) = 0$ , then  $H$  is a torus and  $\partial D_-$  and  $\partial D_+$  are parallel curves on  $H$  (ignoring  $T \cap H$ .) Thus,  $M$  contains a non-separating sphere, contradicting our running assumption. Hence  $g(H) = 2$ .

Since  $\partial D_-$  is non-separating on the genus 2 surface  $H$ , the surface  $H' = H \setminus \partial D_-$  is a genus one surface with two boundary components. If  $\partial D_+$  is non-separating on  $H'$ , then  $\partial D_- \cup \partial D_+$  is non-separating on  $H$ , and we are done. Thus, we may assume that  $\partial D_+$  separates  $H$ . Together with components of  $\partial H'$ , the curve  $\partial D_+$  must either bound a disc, a pair-of-pants, or an annulus in  $H'$ . Since  $\partial D_+$  is non-separating on  $H$ , we can rule out the first two possibilities. The third possibility implies that  $\partial D_+$  is parallel in  $H$  to  $\partial D_-$  and so again  $M$  contains a non-separating sphere, a contradicting the running hypothesis again. Thus, the conclusion holds if  $g(H) \leq 2$ .

Suppose, therefore, that no closed surface in  $M$  separates and that  $\partial D_-$  and  $\partial D_+$  are both non-separating on  $H$ , but  $\partial D_- \cup \partial D_+$  is separating. Let  $F_1$  and  $F_2$  be the two components of  $H \setminus (\partial D_- \cup \partial D_+)$ . Since  $D_-$  is non-separating in the v.p.-compressionbody  $H_\downarrow$  below  $H$ , there exists a properly-embedded arc  $\psi_-$  in  $H_-$  joining  $F_1$  to  $F_2$  which is disjoint from  $D_-$ . Likewise, there is a properly-embedded arc  $\psi_+$  in the v.p.-compressionbody  $H_\uparrow$  above  $H$  which joins  $F_1$  and  $F_2$  and is disjoint from  $D_+$ . Since  $F_1$  and  $F_2$  are each path-connected, without loss of generality, the endpoints of  $\psi_-$  and  $\psi_+$  coincide. Then  $\psi_- \cup \psi_+$  is a loop intersecting each of  $F_1$  and  $F_2$  exactly once. Thus, each of the components of the thin surface obtained by untelescoping  $H$  using  $D_-$  and  $D_+$  are non-separating, a contradiction. Thus, the conclusion holds in this case also.  $\square$

#### 4. MINIMALITY RESULTS

In this section we show that both net extent and width (at least under the width hypothesis) are non-negative and detect the unknot. The results and techniques of this section are often applicable more generally - for instance in studying links or graphs of small net extent.

We begin by confirming that the net euler characteristic of a linear multiple v.p.-bridge surface provides an upper bound on the negative euler characteristic of its components. We use the fact that generalized Heegaard splittings (in the sense of Scharlemann-Thompson [15]) can be amalgamated to create a Heegaard surface, a result due to Schultens [19]. We defer to Schultens' paper for a precise definition of amalgamation. (See also [7].)

**Lemma 4.1.** *Assume that  $\mathcal{H} \in \mathbb{H}(M, T)$ . If  $\text{net } \chi(\mathcal{H}) = x \in \mathbb{Z}$ , then every component  $S \sqsubset \mathcal{H}$  has  $-\chi(S) \leq x$ .*

*Proof.* As  $\text{net } \chi(\mathcal{H})$  is computed without taking  $T$  into account, we may ignore  $T$  for the purposes of this proof. Cap off all 2-sphere boundary components of  $\partial M$  with 3-balls, and consolidate parallel thick and thin surfaces in  $\mathcal{H}$  as much as possible to obtain a multiple v.p.-bridge surface  $\mathcal{J}$  for  $M$ . Observe that  $\text{net } \chi(\mathcal{J}) = \text{net } \chi(\mathcal{H}) = x$ . Since we are ignoring  $T$ , we may amalgamate  $\mathcal{J}$  to a Heegaard surface  $J$  for  $M$ . It is straightforward to verify that  $-\chi(J) = x$ . Each component of  $\mathcal{J}$  is the component of a surface  $J'$  obtained by a sequence of compressions of  $J$ . Thus,  $-\chi(J') \leq x$ . Hence, any component  $S$  of  $\mathcal{H}$  that is not consolidated away in the creation of  $\mathcal{J}$  has  $-\chi(S) \leq x$ . Now reconstruct  $\mathcal{H}$  from  $\mathcal{J}$  by inserting in parallel thick and thin surfaces. Suppose that  $F_1$  and  $H_1$  are the first pair of parallel thick and thin surfaces inserted into  $M \setminus \mathcal{J}$  (i.e. the last pair consolidated from  $\mathcal{H}$ ). Let  $H$  be the component of  $\mathcal{J}^+$  which is adjacent to  $F_1$  in  $M \setminus (\mathcal{J} \cup H_1 \cup F_1)$ . By the definition of v.p.-compressionbody,  $-\chi(H) \geq -\chi(F_1)$ , since there is a v.p.-compressionbody such that  $H = \partial_+ C$  and  $F_1 \subset \partial_- C$ . Consequently,  $-\chi(H_1) = -\chi(F_1) \leq -\chi(H) \leq x$ . Proceeding inductively, we show that reinserting all pairs of parallel thin and thick surfaces we see that every component  $S$  of  $\mathcal{H}$  has  $-\chi(S) \leq x$ .  $\square$

**4.1. Compressionbodies.** In this subsection, we determine various inequalities for v.p.-compressionbodies. In future sections we will assemble these to study net extent and width.

For a v.p.-compressionbody  $(C, T_C)$  with  $T_C$  a 1-manifold, define

$$\delta(C, T_C) = \text{ext}(\partial_+ C) - \text{ext}(\partial_- C)$$

Note that  $\delta(B^3, \emptyset) = -1$  and if  $(C, T_C)$  is any other trivial compressionbody or  $(S^1 \times D^2, \emptyset)$  or  $(S^1 \times D^2, \text{core loop})$  then  $\delta(C, T_C) = 0$ .

**Lemma 4.2.** *Suppose that  $(C, T_C)$  is a v.p.-compressionbody other than  $(B^3, \emptyset)$  or  $(S^1 \times D^2, \emptyset)$ . Assume  $T_C$  is a 1-manifold not intersecting any spherical component of  $\partial_- C$  exactly once. Then  $\delta(C, T_C) \geq 0$  and if  $\delta(C, T_C) = 0$  then there is no compressing or semi-compressing disc for  $\partial_+ C$  in  $(C, T_C)$ .*

*Proof.* Suppose that  $(C, T_C)$  is other than  $(B^3, \emptyset)$  or  $(S^1 \times D^2, \emptyset)$ . Let  $\Delta$  be a complete set (possibly empty) of pairwise non-parallel sc-discs for  $(C, T_C)$  such that reducing  $(C, T_C)$  along  $\Delta$  results in the union  $(P, T_P)$  of trivial product compressionbodies. Let  $a$  be the number of compressing and semi-compressing discs in  $\Delta$  and let  $b$  be the number of  $(B^3, \emptyset)$  components of  $(P, T_P)$ . Since  $(C, T_C)$  is not  $(B^3, \emptyset)$  or  $(S^1 \times D^2, \emptyset)$ , each  $(B^3, \emptyset)$  component of  $(P, T_P)$  is adjacent to at least 3 remnants of compressing or semi-compressing discs. Hence,  $2a \geq 3b$ .

Observe that

$$\text{ext}(\partial_+ C) = \text{ext}(\partial_+ P) + a = a - b \geq a/3.$$

Also we have  $\text{ext}(\partial_- C) = \text{ext}(\partial_- P)$  so

$$\delta(C, T_C) \geq a/3 \geq 0.$$

Furthermore, if equality holds, then  $a = 0$  and  $\Delta$  does not contain a compressing or semi-compressing disc. Since this is true for every complete collection  $\Delta$ ,  $\partial_+ C$  does not admit a compressing or semi-compressing disc in  $(C, T_C)$  (Lemma 2.3).  $\square$

The next definition will be useful for analyzing v.p.-compressionbodies.

**Definition 4.3.** Suppose that  $(C, T_C)$  is a v.p.-compressionbody. The **ghost arc graph** for  $(C, T_C)$  is the graph  $G$  whose vertices are the components of  $\partial_- C$  and whose edges are the ghost edges of  $T_C$ .

**Corollary 4.4.** *Suppose that  $(C, T_C)$  is a v.p.-compressionbody. Assume  $T_C$  is a 1-manifold not intersecting any spherical component of  $\partial_- C$  exactly once and that  $\delta(C, T_C) = 0$ . Then  $(C, T_C)$  is one of the following*

- (1)  $(B^3, \text{arc})$  or
- (2)  $(S^1 \times D^2, \emptyset)$ , or
- (3)  $(S^1 \times D^2, \text{core loop})$  or
- (4) *a compressionbody such that every component of  $T_C$  is a vertical arc or ghost arc and  $g(\partial_+ C) = g(\partial_- C) + n - (|\partial_- C| - 1)$ , where  $n$  is the number of ghost arcs in  $T_C$ . Furthermore, the ghost arc graph is connected.*

*Proof.* Assume that  $(C, T_C)$  is not  $(B^3, \text{arc})$ ,  $(S^1 \times D^2, \emptyset)$  or  $(S^1 \times D^2, \text{core loop})$ . Since  $\delta(C, T_C) = 0$ ,  $(C, T_C) \neq (B^3, \emptyset)$ . We may apply Lemma 4.2. A bridge arc in  $T_C$  would imply the existence of a compressing or semi-compressing disc for  $\partial_+ C$ . Similarly, if  $T_C$  contained a closed loop, we would also have a compressing or semi-compressing disc as  $(C, T_C) \neq (S^1 \times D^2, \text{core loop})$ . Thus,  $T_C$  is the union of vertical arcs and ghost arcs.



If  $T_C$  is the (possibly empty) union of vertical arcs, the assumption that  $\delta(C, T_C) = 0$  and  $(C, T_C) \neq (S^1 \times D^2, \emptyset)$  implies that  $(C, T_C)$  is a trivial product compressionbody and the lemma follows. If  $\partial_- C$  is disconnected, there must be ghost arcs joining the components as otherwise  $\partial_+ C$  would have a compressing or semi-compressing disc as  $\partial_+ C$  can be obtained from  $\partial_- C$  by attaching 1-handles together with their cores. Thus  $|\partial_- C| - 1$  ghost arcs are needed to guarantee that  $\partial_+ C$  has no compressing or semi-compressing disc and each ghost arc beyond  $|\partial_- C| - 1$  increases  $g(\partial_+ C)$  by 1.  $\square$

Observe, that by Lemma 4.4, if  $(C, T_C)$  is a v.p.-compressionbody of Type (4), with  $g(\partial_+ C) = g(\partial_- C)$ , then the ghost arc graph is a (possibly empty) tree. If  $T_C$  is non-empty and irreducible, and if the components of  $\partial_- C$  are spheres, then each leaf of the ghost arc graph must be incident to a vertical arc component of  $T_C$ .

We will piece the previous observations together with the following equations. Observe that if  $\mathcal{H}$  is oriented, then since each component of  $\mathcal{H}$  is adjacent to precisely two components of  $(M, T) \setminus \mathcal{H}$ :

$$(1) \quad 2 \text{netext}(\mathcal{H}) - \text{ext}(\partial M) - \sum_v (n_v - 2)/2 = \sum_{(C, T_C)} \delta(C, T_C)$$

where the sum on the left is over all vertices  $v$  in  $T$  and  $n_v$  is the valence of the vertex  $v$  and the sum on the right is over all components  $(C, T_C)$  of  $(M, T) \setminus \mathcal{H}$  after drilling out the vertices of  $T$ .

Similar considerations show that (using the same notation):

$$(2) \quad w(\mathcal{H}) - \sum_{F \sqsubset \partial M} \text{ext}^2(F) - \sum_v (n_v - 2)^2/4 = \sum_{(C, T_C)} \text{ext}^2(\partial_+ C) - \sum_{F \sqsubset \partial_- C} \text{ext}^2(F).$$

**Corollary 4.5** (Non-negativity). *Suppose that  $(M, T)$  is irreducible and that  $\mathcal{H} \in \mathbb{H}(M, T)$ . Assume that no component of  $(M, T) \setminus \mathcal{H}$  is  $(B^3, \emptyset)$ . Then*

$$\text{netext}(\mathcal{H}) \geq \frac{1}{2} (\text{ext}(\partial M) - \chi(T) + n/2) \geq 0$$

and

$$w(\mathcal{H}) \geq \text{netext}(\mathcal{H}).$$

*Proof.* We may assume that we have drilled out vertices of  $T$  so that  $T$  is a 1-manifold. By equation (1), we have:

$$2 \text{netext}(\mathcal{H}) = \text{ext}(\partial M) + \sum_v (n_v - 2)/2 + \sum_{(C, T_C)} \delta(C, T_C)$$

Notice that  $\delta(S^1 \times D^2, \emptyset) = 0$  and recall that no component of  $(M, T) \setminus \mathcal{H}$  is  $(B^3, \emptyset)$ . Since  $T$  is irreducible, by Lemma 4.2,  $\delta(C, T_C) \geq 0$  for all  $(C, T_C) \sqsubset (M, T) \setminus \mathcal{H}$ . Suppose that  $T$  has  $V$  interior vertices,  $E$  edges, and  $n = T \cap \partial M$ . We have

$$2E = n + \sum_v n_v.$$

Thus, recalling that  $-\chi(T) = E - (V + n)$ :

$$\begin{aligned} 2 \text{netext}(\mathcal{H}) &\geq \text{ext}(\partial M) + E - n/2 - V \\ &= \text{ext}(\partial M) - \chi(T) + n/2 \end{aligned}$$

Let  $V'$  be the union of the sphere components of  $\partial M$ . Recall that  $2E \geq 3(V + V')$ . Then, from above,

$$\begin{aligned} \text{ext}(\partial M) - \chi(T) + n/2 &= -\frac{1}{2}\chi(\partial M \setminus V') - |V'| + \frac{n}{2} + E - \frac{n}{2} - V \\ &\geq E - (V + |V'|) \\ &\geq \frac{1}{2}(V + |V'|) \\ &\geq 0. \end{aligned}$$

Observe that equality holds only if  $V + |V'| = 0$  and  $\chi(\partial M) = 0$ . This happens only if  $T$  is either empty or a link and  $\partial M$  is empty or the union of tori.

We now consider width. Recall that for each  $F \sqsubset \mathcal{H}$ ,  $\text{ext}(F) \geq 0$  since  $T$  is irreducible and since no component of  $\mathcal{H}$  is a sphere disjoint from  $T$ . Thus, for any component  $(C, T_C)$  of  $(M, T) \setminus \mathcal{H}$ , we have

$$\begin{aligned} \text{ext}^2(\partial_+ C) - \sum_{F \sqsubset \partial_- C} \text{ext}^2(F) &\geq \text{ext}^2(\partial_+ C) - \text{ext}^2(\partial_- C) \\ &= (\text{ext}(\partial_+ C) + \text{ext}(\partial_- C))\delta(C, T_C). \end{aligned}$$

If  $(\text{ext}(\partial_+ C) + \text{ext}(\partial_- C)) = 0$ , then  $\text{ext}(F) = 0$  for each component  $F$  of  $\partial C$ . This implies that  $\delta(C, T_C) = 0$ . Thus,

$$\text{ext}^2(\partial_+ C) - \sum_{F \sqsubset \partial_- C} \text{ext}^2(F) \geq \delta(C, T_C).$$

By Equation (2) and by Equation (1), we have:

$$\begin{aligned} \sum_{F \sqsubset \partial M} \text{ext}^2(F) + \sum_v (n_v - 2)^2/4 + \sum_{(C, T_C)} \left( \text{ext}^2(\partial_+ C) - \sum_{F \sqsubset \partial_- C} \text{ext}^2(F) \right) &= w(\mathcal{H}) \\ &\geq \sum_{F \sqsubset \partial M} \text{ext}(F) + \sum_v (n_v - 2)/4 + \sum_{(C, T_C)} \delta(C, T_C) \\ &\geq \frac{1}{2} \left( \text{ext}(\partial M) + \sum_v (n_v - 2)/2 + \sum_{(C, T_C)} \delta(C, T_C) \right) = \text{netext}(\mathcal{H}). \end{aligned}$$

Hence,

$$w(\mathcal{H}) \geq \text{netext}(\mathcal{H}),$$

as desired.  $\square$

**Remark 4.6.** Observe from the proof of Corollary 4.5, that if  $\text{netext}(\mathcal{H}) = (\text{ext}(\partial M) - \chi(T) + n/2)/2$  and if  $(C, T_C)$  is obtained from a component of  $(M, T) \setminus \mathcal{H}$  by drilling out vertices, then  $\delta(C, T_C) = 0$ . We will make use of this in the next corollary. Furthermore, it follows from the proof that if  $\text{netext}(\mathcal{H}) = 0$ , then  $T$  is either empty or a link and  $\partial M$  is empty or the union of tori.

**Corollary 4.7.** Assume that  $(M, T)$  is irreducible, with  $T$  a 1-manifold, and is other than  $(S^3, \emptyset)$  or  $(B^3, \emptyset)$ . Suppose that  $\mathcal{H} \in \mathbb{H}(M, T)$  is locally thin with either  $\text{netext}(\mathcal{H})$  or  $w(\mathcal{H})/2$  equal to

$$\frac{1}{2} (\text{ext}(\partial M) - \chi(T) + |T \cap \partial M|/2)$$

Then for every component  $(C, T_C)$  of  $(M, T) \setminus \mathcal{H}$  we have  $\delta(C, T_C) = 0$ . In particular, each  $(C, T_C)$  is one of:

- (1)  $(B^3, \text{arc})$
- (2)  $(S^1 \times D^2, \emptyset)$
- (3)  $(S^1 \times D^2, \text{core loop})$

- (4) a compressionbody such that every component of  $T_C$  is a vertical arc or ghost arc and  $g(\partial_+ C) = g(\partial_- C) + n - (|\partial_- C| - 1)$ , where  $n$  is the number of ghost arcs in  $T_C$ . Furthermore, the ghost arc graph is connected.

*Proof.* By Theorem 2.11, since  $(M, T)$  is irreducible and is not  $(S^3, \emptyset)$  or  $(B^3, \emptyset)$ , no component of  $(M, T) \setminus \mathcal{H}$  is a  $(B^3, \emptyset)$ . We thus satisfy the hypothesis of Corollary 4.5. If we have equality for  $w(\mathcal{H})/2$ , then by that corollary we also have equality for net extent. By Remark 4.6,  $\delta(C, T_C) = 0$  for each  $(C, T_C) \sqsubset (M, T) \setminus \mathcal{H}$ . By Lemma 4.4, the result follows.  $\square$

We minimize net extent and width to show that they are non-negative half-integer valued invariants of (3-manifold, graph) pairs.

**Corollary 4.8.** *Suppose that  $(M, T)$  is an irreducible (3-manifold, graph) pair other than  $(S^3, \emptyset)$ . Assume that no component of  $\partial M$  is a sphere intersecting  $T$  two or fewer times. Let  $x \geq 2g(M) - 2$ . Then*

$$\text{netext}_x(M, T) \geq \frac{\text{ext}(\partial M) - \chi(T)}{2} + |\partial M \cap T|/4 \geq 0.$$

Furthermore, if  $x \leq 2$  or if every closed surface in  $M$  separates, then

$$w_x(M, T) \geq \text{netext}_x(M, T).$$

*Proof.* Let  $H$  be a minimal genus Heegaard surface for  $M$  so that  $-\chi(H) = 2g(M) - 2$ . Isotope  $H$  to be disjoint from the vertices of  $T$ . Drill out the vertices of  $T$  to obtain  $(\mathring{M}, \mathring{T})$  and observe that  $H$  is still a Heegaard surface for  $\mathring{M}$ . It is a standard result (cf. [5, Lemma 2.1]) that  $\mathring{T}$  can be isotoped to intersect the compressionbodies on either side of  $H$  in bridge arcs and vertical arcs. Filling the vertices of  $T$  back in, the surface  $H$  is a multiple v.p.-bridge surface for  $(M, T)$ . Performing generalized destabilizations, unperturbations, consolidations, and undoing removable edges shows that  $\mathbb{H}(M, T) \neq \emptyset$  and that there is an element with net euler characteristic at most  $2g(M) - 2$ . (In fact,  $\text{net } \chi(\mathcal{H}) = 2g(M) - 2$ .) Let  $\mathcal{H} \in \mathbb{H}(M, T)$ . By Theorem 2.10, there exists  $\mathcal{K} \in \mathbb{H}(M, T)$  such that  $\mathcal{H} \rightarrow \mathcal{K}$  and  $\mathcal{K}$  is locally thin. By Lemma 3.3,  $\text{net } \chi(\mathcal{K}) \leq \text{net } \chi(\mathcal{H})$ . By Theorem 2.11, no component of  $\mathcal{H}$  is a 2-sphere disjoint from  $T$ . Thus, no component of  $(M, T) \setminus \mathcal{H}$  is  $(B^3, \emptyset)$ . By Corollary 4.5, we have

$$\text{netext}(\mathcal{H}) \geq \frac{1}{2}(\text{ext}(\partial M) - \chi(T) + n/2) \geq 0.$$

Hence,

$$\text{netext}_x(M, T) \geq \frac{1}{2}(\text{ext}(\partial M) - \chi(T) + n/2) \geq 0,$$

for all  $x \geq 2g(M) - 2$ .

If  $x \leq 2$  or if every closed surface in  $M$  separates, then the width hypothesis holds for  $\mathcal{H}$ , and thus also for  $\mathcal{K}$ . The result then follows as before.  $\square$

The next lemma is essential for classifying knots with low net extent. It can also be used (as we will show in future work) to classify  $\theta$ -graphs with low net extent. For that reason, we give a very general statement. It provides a classification of linear multiple v.p.-bridge surfaces which are a union of spheres and for which all components  $(C, T_C)$  of  $(M, T) \setminus \mathcal{H}$  have  $\delta(C, T_C) = 0$ .

**Lemma 4.9.** *Suppose that  $(M, T)$  is a pair, other than  $(S^3, \emptyset)$ , such that  $T$  is irreducible 1-manifold. Assume that  $|\partial M| \leq 2$ . Suppose that  $\mathcal{H} \in \mathbb{H}(M, T)$  is such that no component of  $(M, T) \setminus \mathcal{H}$  is a trivial product compressionbody adjacent to  $\mathcal{H}^-$ . Suppose also that:*

- (1) Each component of  $\mathcal{H}$  is a sphere, none of which is disjoint from  $T$ .

(2) For all  $(C, T_C) \sqsubset (M, T) \setminus \mathcal{H}$ , we have  $\delta(C, T_C) = 0$

Then one of the following holds:

- (1)  $\mathcal{H}$  is connected,  $(M, T) = (S^3, \text{unknot})$  with  $|T \cap \mathcal{H}| = 2$ .
- (2)  $\mathcal{H}$  is connected,  $(M, T) = (B^3, \text{unknotted arc})$  with  $|T \cap \mathcal{H}| = 2$
- (3)  $M = S^2 \times I$ , and  $T$  can be isotoped to be the union of two fibers.

*Proof.* We begin by collecting some facts concerning a component  $(C, T_C) \sqsubset (M, T) \setminus \mathcal{H}$  other than  $(B^3, \text{arc})$ . By hypothesis,  $\delta(C, T_C) = 0$  for all components of  $(M, T) \setminus \mathcal{H}$ . By the comments following Lemma 4.4, each component  $(C, T_C) \sqsubset (M, T) \setminus \mathcal{H}$  is either a  $(B^3, \text{arc})$  or  $\partial_- C$  is the non-empty union of spheres and the ghost arc graph  $G$  is a tree with each leaf adjacent to vertical arc components of  $T_C$ . In particular, the number of ghost arcs is exactly one less than  $|\partial_- C|$ . Thus,  $|\partial_+ C \cap T|$  is at least the number of leaves of  $G$ . Furthermore, if  $\partial_- C$  had a single component, then  $(C, T_C)$  would be a trivial product compressionbody. The only way this can happen is if  $\partial_- C = \partial M$ . See, for example, Figure 4. Finally, observe that if  $|\partial_+ C \cap T| = 2$ , then  $G$  is a graph homeomorphic to a line segment and  $T_C$  contains exactly two vertical arcs.

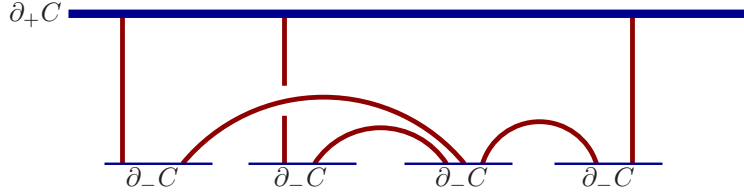


FIGURE 4. An example of the v.p.-compressionbody  $(C, T_C)$  where  $G$  is a tree not homeomorphic to a line segment.

We now embark on the proof.

Suppose, first, that  $\mathcal{H}$  is connected. Since  $|\partial M| \leq 2$  and  $\mathcal{H}$  is a sphere, either  $M = S^3$ ,  $M = B^3$ , or  $M = S^2 \times I$ . Let  $(C, T_C)$  and  $(E, T_E)$  be the two components of  $(M, T) \setminus \mathcal{H}$ . If  $M = S^3$ , then each must be a  $(B^3, \text{arc})$  and so  $T$  is the unknot and  $|\mathcal{H} \cap T| = 2$ . If  $M = B^3$ , then, without loss of generality, we may assume that  $\partial M \subset \partial_- C$ . In this case  $(E, T_E) = (B^3, \text{arc})$  and  $(C, T_C)$  must be a trivial product compressionbody. Consequently,  $T$  is an unknotted arc intersecting  $\mathcal{H}$  twice. Suppose, therefore, that  $M = S^2 \times I$ . Consider first the case when  $\partial_- C$  or  $\partial_- E$  is empty. Without loss of generality,  $\partial_- C = \emptyset$ . In this case,  $(C, T_C) = (B^3, \text{arc})$  and  $|\partial_- E| = 2$ . Since  $|\partial_+ E \cap T| = 2$ ,  $T_E$  must consist of two vertical arcs and a ghost arc. Now consider the case when both  $\partial_- C$  and  $\partial_- E$  are non-empty. In this case, both  $T_C$  and  $T_E$  are the union of vertical arcs and so each arc of  $T$  intersects  $\mathcal{H}$  exactly once. Thus, if  $\mathcal{H}$  is connected, the lemma holds.

Suppose that  $\mathcal{H}$  is disconnected. In particular,  $\mathcal{H}^- \neq \emptyset$ .

**Case 1:** There is a component  $P_0 \subset \mathcal{H}^-$  such that  $P_0$  does not separate the components of  $\partial M$ .

Let  $(M_0, T_0)$  be the component of  $(M, T) \setminus P_0$  which does not contain any component of  $\partial M$ . Note that we may consider  $\partial M_0 = P_0$ . Let  $\mathcal{H}_0$  be the components of  $\mathcal{H}$  in the interior of  $M_0$ , so that  $\mathcal{H}_0$  is a multiple v.p.-bridge surface for  $(M_0, T_0)$ . Let  $(A_0, T_0)$  be the component of  $(M_0, T_0) \setminus \mathcal{H}_0$  such that  $P_0 \subset \partial_- A_0$ . See Figure 5.

Since  $(A_0, T_0)$  is not a trivial product v.p.-compressionbody,  $\partial_- A_0$  is disconnected. By our choice of  $M_0$ ,  $\partial_- A_0 \subset \mathcal{H}$ . Let  $P_1$  be a component of  $\partial_- A_0 \setminus P_0$ . Observe that  $P_1$  does not separate the components of  $\partial M$ . Let  $(A_1, T_1) \neq (A_0, T_0)$  be the component of  $(M_0, T_0) \setminus \mathcal{H}_0$  such that  $P_1 \subset \partial_- A_1$ . As with  $A_0$ , we observe that  $\partial_- A_1$  is disconnected and is a subset of  $\mathcal{H}^-$ . Let  $P_2$  be a

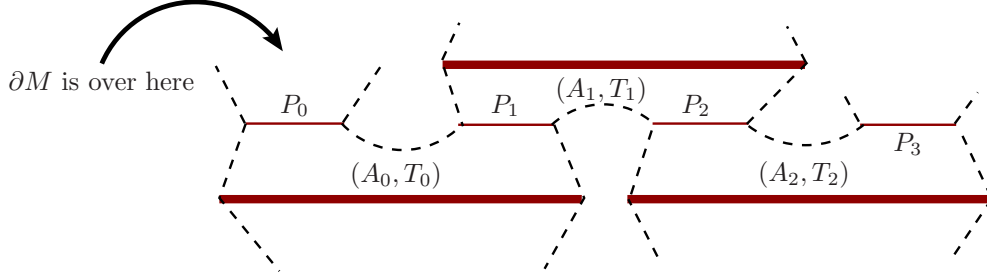


FIGURE 5. The configuration from Case 1 of the proof of Lemma 4.9. The graph  $T$  has been omitted from the diagram for clarity.

component of  $\partial_- A_1$  other than  $P_0$ . Since each component of  $\mathcal{H}$  separates  $M$ ,  $P_2$  is not a component of  $\partial_- A_0$ . Continuing in a similar vein, we construct an infinite sequence  $P_0, P_1, P_2, \dots$  of distinct components of  $\mathcal{H}^-$ . This contradicts the fact that  $\mathcal{H}$  has only finitely many components. Thus this case cannot occur.

**Case 2:** Every component of  $\mathcal{H}^-$  separates the components of  $\partial M$ .

In this case,  $|\partial M| = 2$  and the components of  $\mathcal{H}^-$  are nested. Let the components of  $\partial M \cup \mathcal{H}^-$  be numbered  $P_0, P_1, P_2, \dots, P_n$  so that  $P_0$  and  $P_n$  are the components of  $\partial M$  and so that  $P_i$  and  $P_{i+1}$  cobound a 3-dimensional submanifold of  $M$  containing no component of  $\mathcal{H}^-$  in its interior for all  $i \in \{0, \dots, n-1\}$ .

Let  $i \in \{1, \dots, n-1\}$ , so that  $P_i \subset \mathcal{H}^-$ . Let  $H_-$  and  $H_+$  be the components of  $\mathcal{H}^+$  on either side of  $P_i$ . Let  $(C_1, T_1), (C_2, T_2), (C_3, T_3)$ , and  $(C_4, T_4)$  be the components of  $(M, T) \setminus \mathcal{H}$  such that  $P_i \subset \partial_- C_2 \cap \partial_- C_3$ . See Figure 6. Since no  $(C_j, T_j)$  is a trivial product compressionbody adjacent to  $\mathcal{H}^-$ , both  $(C_2, T_2)$  and  $(C_3, T_3)$  contain a component of  $\partial M \cup \mathcal{H}^-$  other than  $P_i$ . Since the  $P_j$  are nested, the v.p.-compressionbodies  $(C_1, T_1)$  and  $(C_2, T_2)$  are both disjoint from both  $\partial M$  and  $\mathcal{H}^-$ . Consequently, they are both  $(B^3, \text{arc})$ . Also, both of  $T_2$  and  $T_3$  are the union of a ghost arc and two vertical arcs.

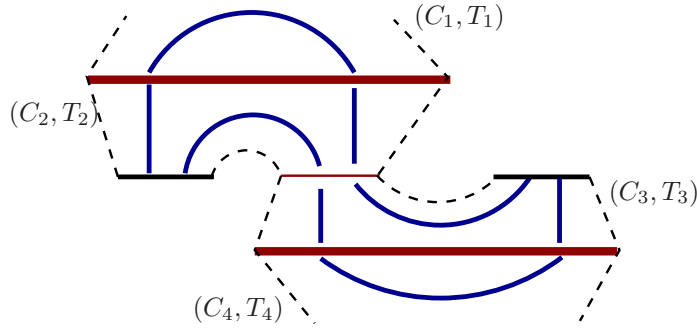


FIGURE 6. The configuration from Case 2 of the proof of Lemma 4.9

The submanifold  $M_i$  between  $P_{i-1}$  and  $P_{i+1}$  is then easily seen to be homeomorphic to  $S^2 \times I$  with  $T \cap M_i$  the union of two parallel arcs each joining  $P_{i-1}$  to  $P_{i+1}$  and each intersecting  $P_i$  exactly once. Furthermore, it is clear that  $T \cap M_i$  can be isotoped so that it is the union of fibers. An easy inductive argument shows that  $M = S^2 \times I$  and that  $T$  can be isotoped to be the union of two fibers.  $\square$

**4.2. Detecting the unknot.** In this subsection we show that net extent and width detect the unknot. For our purposes, a Hopf link in a lens space is the union of the cores of the solid tori on either side of a genus 1 Heegaard surface for the lens space.

**Theorem 4.10** (Detecting the unknot). *Suppose that  $(M, T)$  is an irreducible  $(3\text{-manifold, graph})$  pair with  $T \neq \emptyset$  such that  $(M, T)$  does not have a (lens space, core loop) or  $(S^1 \times D^2, \text{core loop})$  summand. Assume  $T$  has at most one component which is a knot. Assume also that no component of  $\partial M$  is a sphere intersecting  $T$  exactly twice. If  $\text{netext}_x(M, T) = 0$  for some  $x \geq 2g(M) - 2$ , then  $(M, T) = (S^3, \text{unknot})$ . Furthermore, if  $x \leq 2$  or if every closed surface in  $M$  separates, then the same result holds if  $w_x(M, T) = 0$ .*

*Proof.* We will show that the theorem holds for all  $x$  with  $2g(M) - 2 \leq x < \infty$ . By definition, it will then also hold when  $x = \infty$ . We only use the fact that  $T$  has at most one knot component at one point in the argument, so for most of it we work somewhat more generally.

By Theorem 4.8, if  $x \leq 2$  or if  $M$  contains no non-separating closed surface, then  $w_x(M, T) \geq \text{netext}_x(M, T)$ . Consequently, we may assume that  $\text{netext}_x(M, T) = 0$  for some  $x \geq 2g(M) - 2$ .

Let  $\mathcal{H} \in \mathbb{H}(M, T)$  be such that  $\text{net } \chi(\mathcal{H}) \leq x$  and  $\text{netext}(\mathcal{H}) = \text{netext}_x(M, T)$ . By Theorem 2.10 and Corollary 3.3, we may also assume that  $\mathcal{H}$  is locally thin. In particular, by Theorem 2.11, no component of  $\mathcal{H}$  is a sphere disjoint from  $T$  and no component of  $(M, T) \setminus \mathcal{H}$  is a trivial product compressionbody adjacent to a component of  $\mathcal{H}^-$ . Furthermore, by Remark 4.6,  $T$  is a 1-manifold and  $\partial M$  is the (possibly empty) union of tori. Since  $T$  is a 1-manifold,  $\chi(T)$  is the number  $n$  of edges of  $T$  having both endpoints on  $\partial M$ . Hence,  $\text{ext}(\partial M) - \chi(T) = 0$  and so, by Corollary 4.8,  $n = 0$ . Thus,  $T$  is a link. By our hypothesis, this implies that  $T$  is a knot; but we will not use this quite yet.

By Remark 4.6, for all components  $(C, T_C)$  of  $(M, T) \setminus \mathcal{H}$  we have  $\delta(C, T_C) = 0$ . By Corollary 4.4, if  $(C, T_C)$  is a component of  $(M, T) \setminus \mathcal{H}$ , then  $(C, T_C)$  is one of:

- (1)  $(B^3, \text{arc})$ ,
- (2) (solid torus,  $\emptyset$ ),
- (3) (solid torus, core loop),
- (4) v.p.-compressionbody such that every component of  $T_C$  is a vertical arc or ghost arc and there is no compressing disc for  $\partial_+ C$  in the complement of  $T$ .

**Case 1:**  $T$  is disjoint from  $\mathcal{H}$ .

Then,  $\mathcal{H}$  is a multiple v.p.-bridge surface for the exterior of  $T$ . As in the proof of Lemma 4.9, we may amalgamate  $\mathcal{H}$  to a Heegaard surface  $H$  for the exterior of  $T$ . It is easily verified that

$$-\chi(H) = \text{net } \chi(\mathcal{H}) = 2 \text{netext}(\mathcal{H}) = 0.$$

Thus,  $H$  is a torus. Since no component of  $\partial M$  is a sphere intersecting  $T$  exactly twice,  $(M, T)$  is one of:

- $(S^3, \text{unknot})$
- $(S^3, \text{Hopf link})$
- (solid torus, core loop)
- (lens space, core loop)
- (lens space, Hopf link)

□(Case 1)

**Case 2:** Some  $(C, T_C)$  is a (solid torus,  $\emptyset$ ) or (solid torus, core loop).



Let  $(E, T_E) \neq (C, T_C)$  be the other v.p.-compressionbody adjacent to  $\partial_+ C$ . If  $(E, T_E)$  is (solid torus,  $\emptyset$ ) or (solid torus, core loop) then  $\mathcal{H} = \partial_+ E = \partial_+ C$  and we are in Case 1. The v.p.-compressionbody  $(E, T_E)$  must therefore be of type (4). Since  $T \cap \partial_+ C = \emptyset$ ,  $T_E$  must be a (at the moment, potentially empty) union of ghost arcs. Recall that each component of  $\partial_- E$  must have genus 0 or 1 and that there is at most one genus 1 component.

Suppose that  $\partial_- E$  has a genus 1 component. If  $\partial_- E$  is disconnected, then the other components of  $\partial_- E$  (apart from the genus 1 component) are spheres. By Lemma 4.2,  $\partial_+ E$  does not have a compressing or semi-compressing disc. Thus, each sphere component of  $\partial_- E$  is adjacent to a ghost arc. Since  $T$  is irreducible, by Lemma 4.4, there is a sphere component adjacent to a vertical arc, a contradiction. Thus  $\partial_- E$  is connected and is a torus and  $T_E = \emptyset$ . Since  $T \neq \emptyset$ ,  $\partial_- E \subset \mathcal{H}^-$ . Hence,  $(E, T_E)$  is a trivial product compressionbody adjacent to  $\mathcal{H}^-$ . This contradicts the thinness of  $\mathcal{H}$ . Consequently,  $\partial_- E$  is the non-empty union of spheres.

Since  $\partial_+ E$  is incompressible in  $(E, T_E)$ ,  $T_E$  must be non-empty. Thus there is a cut-disc  $E$  for  $\partial_+ E$  in  $(E, T_E)$ . Since  $(C, T_C)$  is either a (solid torus,  $\emptyset$ ) or a (solid torus, core loop) there is also a compressing disc or cut disc  $D$  for  $\partial_+ E = \partial_+ C$  in  $(C, T_C)$ . Now we apply our hypothesis on  $T$ . Since  $T$  is a knot,  $D$  must be a compressing disc. Isotope  $D$  and  $E$  to minimize the number of intersections between their boundaries. If their boundaries are disjoint or intersect exactly once, we contradict the thinness of  $\mathcal{H}$ . (Either  $\partial_+ E$  would be c-weakly reducible or meridionally stabilized.) Let  $P$  be the sphere which results from compressing  $\partial_+ C = \partial_+ E$  using the cut disc. Observe that  $P$  bounds a submanifold  $W$  of  $M$  containing torus  $\partial_+ C = \partial_+ E$ , which is a genus 1 Heegaard surface for  $W$ . Since the boundaries of the cut disc and compressing disc intersect at least twice,  $W$  is a lens space (other than  $S^3$  or  $S^1 \times S^2$ .) Furthermore,  $|P \cap T| = 2$  and  $T \cap W$  is an unknotted arc disjoint from the Heegaard surface  $\partial_+ C = \partial_+ E$ . Hence,  $(M, T)$  has a (lens space, core loop) summand, a contradiction. Consequently, no component of  $(C, T_C)$  is (solid torus,  $\emptyset$ ).

**Case 3:** Each component of  $(M, T) \setminus \mathcal{H}$  is either a trivial ball compressionbody or a v.p.-compressionbody of Type (4) above.

**Case 3a:** No component of  $\mathcal{H}$  has positive genus.

In this case, each component of  $\mathcal{H}$  is a sphere. By Lemma 4.9,  $\mathcal{H}$  is connected, and  $(M, T) = (S^3, \text{unknot})$  with  $|T \cap \mathcal{H}| = 2$ .

**Case 3b:** Some component of  $\mathcal{H}$  has positive genus.

Since  $\mathcal{H}$  is linear, each component of  $\mathcal{H}^+$  separates  $M$ . If some component of  $\mathcal{H}^-$  has positive genus then so do both of the adjacent component of  $\mathcal{H}^+$ . Out of all components of  $\mathcal{H}^+$  having positive genus, choose  $H$  to be an innermost (in  $M$ ) such component, i.e.  $H$  (together, perhaps, with some subset of  $\partial M$ ) bounds a 3-manifold  $W \subset M$  such that no component of  $\mathcal{H}^+$  in the interior of  $W$  has positive genus.

Let  $(C, T_C)$  be the component of  $(M, T) \setminus \mathcal{H}$  which is contained in  $W$  such that  $H = \partial_+ C$ . Since  $\delta(C, T_C) = 0$ , the v.p.-compressionbody  $(C, T_C)$  has the property that  $T_C$  is the union of ghost arcs and vertical arcs and  $H$  has no compressing disc in  $C$ . Suppose, for the time being, that  $\partial_- C \cap \partial M \neq \emptyset$ , then since  $T \cap \partial M = \emptyset$ , the v.p.-compressionbody  $(C, T_C)$  is a trivial product compressionbody disjoint from  $T$ . Let  $(E, T_E)$  be the v.p.-compressionbody other than  $(C, T_C)$  such that  $\partial_+ E = \partial_+ C$ . If  $T_E = \emptyset$ , then by Corollary 4.7,  $(E, T_E)$  is either  $(S^1 \times D^2, \emptyset)$  or a trivial product compressionbody. It cannot be the latter as  $T \neq \emptyset$  and no component of  $(M, T) \setminus \mathcal{H}$  is a trivial product compressionbody adjacent to a component of  $\mathcal{H}^-$ . Thus,  $T_E$  is the non-empty union of ghost arcs and each component of  $\partial_- E$  is a sphere. Additionally, the number of ghost arcs

is equal to  $|\partial_- E|$  and so it is straightforward to verify that  $(M, T)$  has a (solid torus, core loop) summand, a contradiction. Thus,  $\partial_- C \cap \partial M = \emptyset$ .

By our choice of  $H$ , the surface  $\partial_- C$  is the union of spheres. Since each sphere in  $M$  separates, a component  $P \subset \partial_- C$  bounds a 3-manifold  $W' \subset W$  such that  $\mathcal{H} \cap W'$  is the union of spheres and  $\partial W' = P$ . By Lemma 4.9,  $(W', T \cap W') = (B^3, \text{unknotted arc})$  and  $|T \cap \mathcal{H}^+ \cap W'| = 2$ . This implies that some component of  $(M, T) \setminus \mathcal{H}$  adjacent to  $P$  is a trivial product compressionbody, contradicting the thinness of  $\mathcal{H}$ . Thus, no component of  $\mathcal{H}$  has positive genus.  $\square$

**Remark 4.11.** The minimal bridge sphere for the unknot is a v.p.-bridge surface for the unknot, showing that  $\text{netext}_2(S^3, \text{unknot}) = 0$ . Observe that the Hopf Link  $L$  has  $\text{netext}_0(L) = 0$  but is not the unknot (namely take the Heegaard surface for  $S^3$  containing the components of  $L$  as core loops for the solid tori on either side). Likewise, the core of a solid torus or the core of one of the solid tori in a genus 1 Heegaard splitting of the lens space will also have  $\text{netext}_0(T) = 0$  but is not the unknot (i.e. does not bound a disc). Finally, as we will see, net extent and width are additive under connected sum and so taking the connected sum with (lens space, core loop) or (solid torus, core loop) will not change net extent or width.

## 5. ADDITIVITY OF NET EXTENT AND WIDTH

In this section we prove that net extent and width satisfy an additivity property with respect to connected sum and trivalent vertex sum. In fact, apart from some hypotheses on  $M$  and  $T$ , the only properties of net extent and width that we use are that they are order preserving with respect to extended thinning sequences and that they depend on euler characteristic and the number of intersections with  $T$ . For convenience, therefore, and with a view to the fact that there are other invariants which have similar properties we prove our additivity theorem in a rather abstract setting. Theorem 5.4 shows that super-additivity holds; Theorem 5.5 shows that sub-additivity holds; and Theorem 5.7 puts those together to show the additivity result for net extent and width.

We begin by relating the thin levels of a locally thin linear multiple v.p.-bridge surfaces to a prime decomposition.

**Proposition 5.1.** *Assume that  $(M, T)$  is non-trivial and that  $T$  is irreducible. Suppose that  $\mathcal{H} \in \mathbb{H}(M, T)$  is locally thin. Then there exists a subset  $\mathcal{P} \subset \mathcal{H}^-$  such that  $\mathcal{P}$  is the union of decomposing spheres giving a prime decomposition of  $(M, T)$ .*

*Proof.* If  $(M, T)$  is prime, then it is its own prime decomposition and setting  $\mathcal{P} = \emptyset$ , we are done. Assume that  $(M, T)$  contains at least one essential twice or thrice punctured sphere.

Let  $\mathcal{Q}$  be the union of all the twice and thrice-punctured spheres in  $\mathcal{H}^-$ . Let  $(M_i, T_i)$  be a component of  $(M, T) \setminus \mathcal{Q}$ . Let  $(\widehat{M}_i, \widehat{T}_i)$  be the result of capping off twice punctured spheres (corresponding to copies of components of  $\mathcal{Q}$ ) in  $\partial M_i$  with a trivial  $(B^3, \text{arc})$  and capping off thrice punctured spheres (corresponding to copies of components of  $\mathcal{Q}$ ) of  $F$  in  $\partial M_i$  with a 3-ball containing a boundary parallel tree with a single internal vertex.

We claim that  $(\widehat{M}_i, \widehat{T}_i)$  contains no essential twice or thrice-punctured sphere. Suppose, to the contrary, that such a sphere  $F$  exists. Since each component of  $\mathcal{Q}$  is essential in  $(M, T)$ , the surface  $F$  is also essential in  $(M_i, T_i)$ . It is easy to see that the intersection  $\mathcal{H}_i$  of  $\mathcal{H}$  with the interior of  $M_i$  is still a locally thin, linear, multiple v.p.-bridge surface for  $(M_i, T_i)$ . Thus, by Theorem 2.11, there exists an essential twice or thrice-punctured sphere  $P \sqsubset \mathcal{H}_i^-$ . However,  $\mathcal{H}_i^- \subset \mathcal{H}^-$  does not contain any essential twice or thrice-punctured spheres by the definition of  $\mathcal{Q}$  and  $\mathcal{H}_i$ .

Suppose that  $(M_i, T_i)$  has the property that  $(\widehat{M}_i, \widehat{T}_i)$  is  $(S^3, \text{unknot})$ . Since each component of  $\mathcal{Q}$  is essential in  $(M, T)$ ,  $\partial M_i$  has multiple components, each a sphere intersecting  $T$  twice. Let  $P$  be one such component, and let  $(M_j, T_j)$  be the component of  $(M, T) \setminus \mathcal{Q}$  adjacent to  $P$  and not equal to  $(M_i, T_i)$ . Let  $(M', T') = (M_i, T_i) \cup (M_j, T_j)$  and let  $(\widehat{M}', \widehat{T}')$  be the result of capping off components of  $\partial M'$  corresponding to components of  $\mathcal{Q} \setminus P$ . Then  $(\widehat{M}', \widehat{T}')$  is the connected sum of  $(\widehat{M}_j, \widehat{T}_j)$  with  $(S^3, \text{unknot})$ . It is thus homeomorphic to  $(\widehat{M}_j, \widehat{T}_j)$  and so does not contain an essential twice or thrice-punctured sphere. Continuing on in this vein, we may remove some number of components from  $\mathcal{Q}$  to obtain  $\mathcal{P}'$  such that if  $(\widehat{M}', \widehat{T}')$  is obtained by capping off components of  $\partial M'$  corresponding to components of  $\mathcal{P}'$ , where  $(M', T') \subset (M, T) \setminus \mathcal{P}'$  then  $(\widehat{M}', \widehat{T}')$  neither contains an essential twice or thrice-punctured sphere nor is  $(S^3, \text{unknot})$ .

Suppose now that  $(M_i, T_i)$  has the property that  $(\widehat{M}_i, \widehat{T}_i)$  is  $(S^3, \widehat{T}_i)$  with  $\widehat{T}_i$  a trivial  $\theta$ -graph and with some component  $P$  of  $\partial M_i$  a thrice-punctured sphere. Let  $(M_j, T_j) \subset (M, T) \setminus \mathcal{P}'$  be on the other side of  $P$  from  $(M_i, T_i)$ . Let  $(M', T') = (M_i, T_i) \cup (M_j, T_j)$  and let  $(\widehat{M}', \widehat{T}')$  be the result of capping off components of  $\partial M'$  corresponding to components of  $\mathcal{P}' \setminus P$ . Then  $(\widehat{M}', \widehat{T}')$  is the trivalent vertex sum of  $(\widehat{M}_j, \widehat{T}_j)$  with  $(S^3, G)$ , where  $G$  is a trivial  $\theta$ -graph. It is thus homeomorphic to  $(\widehat{M}_j, \widehat{T}_j)$  and so does not contain an essential twice or thrice-punctured sphere. Continuing on in this vein, we can remove some number of components from  $\mathcal{P}'$  to arrive at  $\mathcal{P}$ , the union of some components of  $\mathcal{P}'$ , which give a prime decomposition of  $(M, T)$ .  $\square$

Let  $\mathbb{M}$  be a non-empty set whose elements are irreducible  $(3\text{-manifold, graph})$  pairs  $(M, T)$  with  $M$  connected such that every sphere in  $M$  separates, and  $T$  intersects each sphere in  $\partial M$  at least three times. Suppose that  $\mathbb{M}$  has the property that if  $(M, T) \in \mathbb{M}$  and  $M = (M_1, T_1) \# (M_2, T_2)$  or  $M = (M_1, T_1) \#_3 (M_2, T_2)$  then both  $(M_1, T_1)$  and  $(M_2, T_2)$  are also elements of  $\mathbb{M}$ . Let  $\mathbb{S}$  denote the set of closed surfaces  $S \subset (M, T)$  for some  $(M, T) \in \mathbb{M}$  and let  $\mathbb{S}_0 \subset \mathbb{S}$  be the subset of connected surfaces. Let  $X = (\mathbb{Z} \times \mathbb{Z}) \cap ([-2, \infty) \times [0, \infty))$  and let  $r: \mathbb{S}_0 \rightarrow \mathbb{R}$  be any function which factors through the function  $\mathbb{S}_0 \rightarrow X$  defined by  $S \mapsto (-\chi(S), |T \cap S|)$ . (That is,  $r$  depends only on  $-\chi(S)$  and  $|S \cap T|$ .) Extend  $r$  to a function  $r: \mathbb{S} \rightarrow \mathbb{R}$  linearly, that is if  $S_1, S_2 \in \mathbb{S}_0$  are disjoint then  $r(S_1 \cup S_2) = r(S_1) + r(S_2)$ .

**Example 5.2.** If  $S \subset \mathbb{S}$  and  $k \in \mathbb{N}$ , then the function  $r(S) = \sum_i \text{ext}^k(S_i)$  where the sum is over all the components  $S_i$  of  $S$  is such a function.

For  $(M, T) \in \mathbb{M}$ , recall that  $\mathbb{H}(M, T)$  is the set of reduced, linear, multiple v.p.-bridge surfaces for  $(M, T)$ . Let  $\mathbb{H} = \bigcup_{(M, T) \in \mathbb{M}} \mathbb{H}(M, T)$ . Define  $\text{net } r: \mathbb{H} \rightarrow \mathbb{R}$  by

$$\text{net } r(\mathcal{H}) = r(\mathcal{H}^+) - r(\mathcal{H}^-).$$

If  $x \geq 2g(M) - 2$  and  $(M, T) \in \mathbb{M}$ , we say that  $x$  is **realizable** for  $(M, T)$ . If  $x$  is realizable for  $(M, T)$ , let  $\text{net } r_x(M, T) = \min_{\mathcal{H}} \text{net } r(\mathcal{H})$  where the minimum is over all  $\mathcal{H} \in \mathbb{H}(M, T)$  such that  $\text{net } \chi(\mathcal{H}) \leq x$ . We say that  $r$  is **order-preserving** on  $\mathbb{H}$  if whenever  $\mathcal{H}, \mathcal{K} \in \mathbb{H}$  and  $\mathcal{H} \rightarrow \mathcal{K}$ , then  $\text{net } r(\mathcal{H}) \geq \text{net } r(\mathcal{K})$ . Let  $r_2$  be the value of  $r$  on a sphere twice punctured by  $T$  and  $r_3$  be the value of  $r$  on a sphere thrice-punctured by  $T$ .

**Example 5.3.** If we choose  $\mathbb{M}$  to be the set of all irreducible  $(M, T)$  (with  $M$  connected) and  $r: \mathbb{S}_0 \rightarrow \mathbb{R}$  to be  $r(S) = \text{ext}(S)$ , then  $\text{net } r = \text{netext}$  and  $\text{net } r_x(M, T) = \text{netext}(M, T)$ . In this case,  $r_2 = 0$  and  $r_3 = 1/2$ . Likewise, if we choose  $\mathbb{M}$  to be the set of all irreducible  $(M, T)$  such that  $g(M) \leq 2$  or  $M$  contains no closed non-separating surface and  $r: \mathbb{S}_0 \rightarrow \mathbb{R}$  to be  $r(S) = 2 \text{ext}^2(S)$ , then  $\text{net } r = w$  and  $\text{net } r_x(M, T) = w_x(M, T)$ . In this case also,  $r_2 = 0$  and  $r_3 = 1/2$ .

Return to the general situation, where  $\mathbb{M}$  is any non-empty set whose elements are irreducible (3-manifold, graph) pairs  $(M, T)$  such that  $M$  is connected, every sphere in  $M$  separates,  $T$  intersects each spherical component of  $\partial M$  at least three times and which is closed under taking factors of connected sum and trivalent vertex sum.

**Theorem 5.4** (Super-additivity). *Suppose that  $(M, T) \in \mathbb{M}$  is non-trivial and that  $x$  is realizable for  $(M, T)$  and that  $r$  is order-preserving on  $\mathbb{H}$ . Then there is a prime factorization of  $(M, T)$  into  $(\widehat{M}_1, \widehat{T}_1), \dots, (\widehat{M}_n, \widehat{T}_n)$  and so that there exist integers  $x_1, \dots, x_n$  summing to at most  $x - 2(n - 1)$  so that  $x_i$  is realizable for  $(\widehat{M}_i, \widehat{T}_i)$  and*

$$\text{net } r_x(M, T) \geq -p_2 r_2 - p_3 r_3 + \sum_{i=1}^n \text{net } r_{x_i}(\widehat{M}_i, \widehat{T}_i).$$

where  $p_2$  is the number of connected sums and  $p_3$  is the number of trivalent vertex sums in the decomposition.

*Proof.* Let  $(M, T) \in \mathbb{M}$ . Since  $x$  is realizable, there exists  $\mathcal{H} \in \mathbb{H}(M, T)$  such that  $\text{net } \chi(\mathcal{H}) \leq x$  and  $\text{net } r_x(M, T) = \text{net } r(\mathcal{H})$ . Let  $\mathcal{K}$  be locally thin so that  $\mathcal{H} \rightarrow \mathcal{K}$ . By Theorem 2.10, such a  $\mathcal{K}$  exists and is linear. Since  $r$  is order-preserving,  $\text{net } r(\mathcal{K}) \leq \text{net } r(\mathcal{H})$ . By Lemma 3.3,

$$\text{net } \chi(\mathcal{K}) \leq \text{net } \chi(\mathcal{H}) \leq x.$$

Thus, by our choice of  $\mathcal{H}$ ,  $\text{net } r(\mathcal{K}) = \text{net } r(\mathcal{H})$ .

By Proposition 5.1, there is a subset  $\mathcal{P} \subset \mathcal{K}^-$  which is the union of components such that  $\mathcal{P}$  are the summing spheres giving a prime decomposition of  $(M, T)$ . Split  $(M, T)$  open along  $\mathcal{P}$  to obtain components  $(M_1, T_1), \dots, (M_n, T_n) \sqsubset (M, T) \setminus \mathcal{P}$ . Let  $(\widehat{M}_i, \widehat{T}_i)$  be the result of capping off the components of  $\mathcal{P}$  in  $\partial M_i$  with trivial ball compressionbodies, so that  $(\widehat{M}_1, \widehat{T}_1), \dots, (\widehat{M}_n, \widehat{T}_n)$  give a prime decomposition of  $(M, T)$ . Let  $p_2$  be the number of twice-punctured spheres in  $\mathcal{P}$ . Let  $p_3$  be the number of thrice-punctured spheres in  $\mathcal{P}$ . Observe that  $p_2 + p_3 = n - 1$ . Let  $x_i = \text{net } \chi(\mathcal{K}_i)$ . Note that  $x_i$  is realizable for  $(\widehat{M}_i, \widehat{T}_i)$ . Furthermore,

$$\sum x_i = \text{net } \chi(\mathcal{K}) - 2(n - 1) \leq x - 2(n - 1),$$

since spheres have Euler characteristic equal to 2. Consequently, we have  $x_i$  summing to at most  $x - 2(n - 1)$  and a prime decomposition  $(\widehat{M}_i, \widehat{T}_i)$  of  $(M, T)$  so that  $\text{net } r_{x_i}(\widehat{M}_i, \widehat{T}_i) \leq \text{net } r(\mathcal{K}_i)$ . Notice that:

$$\text{net } r_x(M, T) \geq \text{net } r(\mathcal{K}) = \sum_{H \sqsubset \mathcal{H}^+} r(H) - \sum_{F \sqsubset \mathcal{H}^- \setminus \mathcal{P}} r(S) - p_2 r_2 - p_3 r_3.$$

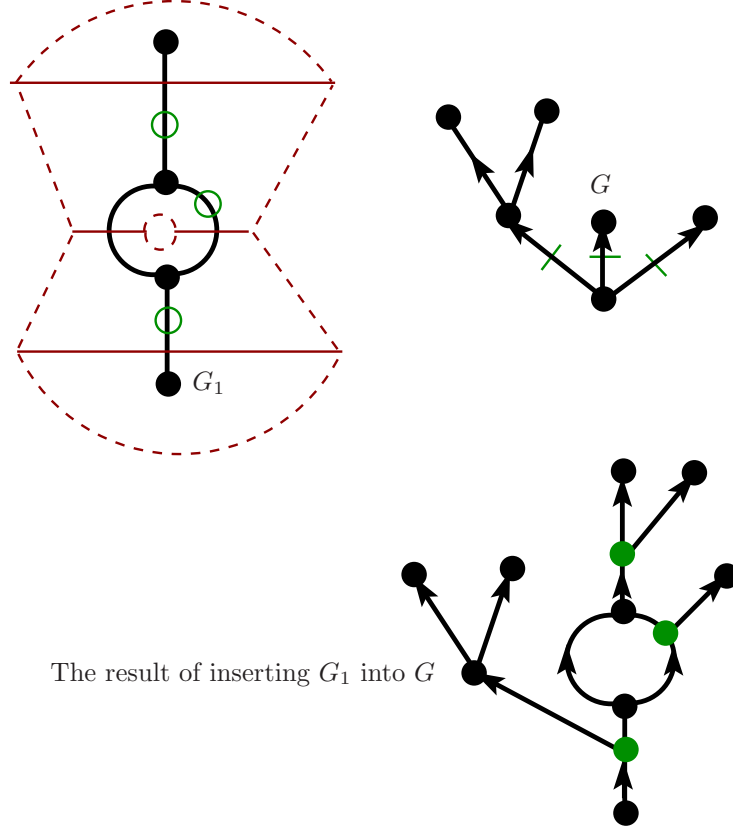
Splitting up the sums according to which  $(M_i, T_i)$  contains the surface shows that

$$\text{net } r_x(M, T) \geq \sum_{i=1}^n \text{net } r(\mathcal{K}_i) - p_2 r_2 - p_3 r_3 \geq \sum_{i=1}^n \text{net } r_{x_i}(\widehat{M}_i, \widehat{T}_i) - p_2 r_2 - p_3 r_3.$$

□

**Theorem 5.5** (Sub-additivity). *Let  $(M, T) \in \mathbb{M}$  is non-trivial. Suppose that  $(M, T)$  is the connected sum and trivalent vertex sum of non-trivial pairs  $(\widehat{M}_1, \widehat{T}_1), \dots, (\widehat{M}_n, \widehat{T}_n)$ , that  $x$  is realizable for  $(M, T)$  and so that there are integers  $x_1, \dots, x_n$  such that each  $x_i$  is realizable for  $(\widehat{M}_i, \widehat{T}_i)$  and  $x_1 + \dots + x_n \leq x - 2(n - 1)$ . Then*

$$\text{net } r_x(M, T) \leq \sum_{i=1}^n \text{net } r_{x_i}(\widehat{M}_i, \widehat{T}_i) - p_2 r_2 - p_3 r_3,$$



The result of inserting  $G_1$  into  $G$

FIGURE 7. The first step of turning  $\mathcal{H}$  into a linear, oriented v.p.-bridge surface. We insert the graph  $G_1$  into the tree  $G$  at the root, ensuring that the orientations of the edges are consistent. The green circles, lines, and dots indicate the points  $p_1$  and the spheres  $P_1$ .

*Proof.* Recall that, by the definition of  $\mathbb{M}$ , each  $(\widehat{M}_i, \widehat{T}_i) \in \mathbb{M}$ . For each  $i \in \{1, \dots, n\}$ , let  $p_i \subset T_i$  be the union of points where  $(\widehat{M}_i, \widehat{T}_i)$  is summed to one of the other (3-manifold, graph)-pairs. For each  $i$ ,  $|p_i| \geq 1$ . The graph in  $M$  dual to the summing spheres is a finite tree. All finite trees have one more vertex than edge. Hence, we have  $\sum_{i=1}^n |p_i| = 2(n-1)$ . For each  $i$ , let  $\mathcal{H}_i \in \mathbb{H}(\widehat{M}_i, \widehat{T}_i)$  be such that  $\text{net } \chi(\mathcal{H}_i) \leq x_i$  and  $\text{net } r(\mathcal{H}_i) = \text{net } r_{x_i}(\mathcal{H}_i)$ .

By general position, we may assume that  $\mathcal{H}_i \cap p_i = \emptyset$ . Let  $f_i: \mathcal{H}_i \rightarrow \{1, \dots, m_i\}$  be the height function for  $\mathcal{H}_i$ . Notice that we can create another multiple v.p.-bridge surface  $\mathcal{H}'_i$  for  $(\widehat{M}_i, \widehat{T}_i)$  by reversing all the transverse orientations and replacing the height  $f(z)$  with a new height  $m_i - f(z) + 1$ . We call this **turning  $\mathcal{H}_i$  upside down**. Turning  $\mathcal{H}_i$  upside down does not change  $\text{net } r(\mathcal{H}_i)$  or  $\text{net } \chi(\mathcal{H}_i)$ .

Let  $(M_i, T_i)$  be the result of removing a small open regular neighborhood of  $p_i$  from  $(\widehat{M}_i, \widehat{T}_i)$ . Let  $P_i$  be the union of the components of  $\partial M_i$  corresponding to the points  $p_i$ . Each component of  $P_i$  is a twice or thrice-punctured sphere. We may view each  $(M_i, T_i)$  as embedded in  $(M, T)$  with  $P_i \subset M$  the union of separating essential twice and thrice-punctured spheres.

Let  $\mathcal{H} = \bigcup_i \mathcal{H}_i \cup P_i$ . Clearly  $\mathcal{H}$  is a v.p.-bridge surface. We will show that, perhaps after turning some of the  $\mathcal{H}_i$  upside down, we can define a transverse orientation and a height function so that  $\mathcal{H}$  is a linear oriented multiple v.p.-bridge surface for  $(M, T)$ .

By the definition of connected sum and trivalent-vertex sum, the graph  $G$  in  $M$  dual to  $\mathcal{H}$  is a tree. Each vertex of  $G$  is some  $(M_i, T_i)$  and we associate the midpoint of each edge of  $G$  with some component  $P$  of some  $P_i$ . Let  $(M_1, T_1)$  be the root of  $G$  and put a partial order  $\leq$  on the vertices of  $G$  so that  $(M_1, T_1)$  is the least element of the partial order and if a vertex  $c$  separates vertices  $a$  and  $b$  then  $a < c < b$ . Orient the edges of  $G$  so that if vertices  $v$  and  $w$  are the endpoints of an edge pointing from  $v$  to  $w$  then  $v < w$ .

Let  $G_i$  be the graph in  $(M_i, T_i)$  dual to  $\mathcal{H}_i$ . The transverse orientation on  $\mathcal{H}_i$  induces an orientation on the edges of  $G_i$ . Suppose that  $P \subset P_i$  is a component. For each  $i \in \{1, \dots, n\}$ , replace the vertex  $(M_i, T_i)$  in  $G$  with the graph  $G_i$ ; we obtain the graph  $G''$  dual to  $\mathcal{H}$ . See Figure 7. Since  $G$  was a tree, after turning  $\mathcal{H}_i$  upside down, if necessary,  $G''$  becomes an oriented graph, inducing transverse orientations on the components of  $P_i$  for each  $i$  and  $\mathcal{H}$  becomes an oriented multiple v.p.-bridge surface. Let  $Q$  be the set of midpoints of edges of  $G$ . These naturally correspond to the components of  $\mathcal{H}$ . The orientations on the edges of  $G$  induce a partial order on  $Q$ . It is then straightforward to define a bijective, order-preserving height function  $h: Q \rightarrow \{1, \dots, k\}$  for some  $k \in \mathbb{N}$ . The surface  $\mathcal{H}$  is thus a linear multiple v.p.-bridge surface for  $T$ . Since each  $\mathcal{H}_i$  was reduced, it is not difficult to see that  $\mathcal{H}$  is reduced.

We now do the necessary calculations to obtain our bound.

Observe that  $x = \text{net } \chi(\mathcal{H}) = \sum_{i=1}^n \text{net } \chi(\mathcal{H}_i) - 2(n-1)$ . Furthermore, since  $r$  depends only on negative euler characteristic and the number of intersections with  $T$ ,

$$\begin{aligned} \text{net } r_x(M, T) \leq \text{net } r(\mathcal{H}) &= \sum_{i=1}^n \text{net } r(\mathcal{H}_i) - p_2 r_2 - p_3 r_3 \\ &= \sum_{i=1}^n \text{net } r(\widehat{M}_i, \widehat{T}_i) - p_2 r_2 - p_3 r_3, \end{aligned}$$

as desired. □

**Corollary 5.6** (Additivity). *Assume that  $r$  is order-preserving and that  $x$  is realizable for some non-trivial  $(M, T) \in \mathbb{M}$ . Then there is a prime factorization of  $(M, T)$  into  $(\widehat{M}_1, \widehat{T}_1), \dots, (\widehat{M}_n, \widehat{T}_n)$  such that there exist integers  $x_1, \dots, x_n$ , summing to at most  $x - 2(n-1)$ , so that  $x_i$  is realizable for  $(\widehat{M}_i, \widehat{T}_i)$ , and*

$$\text{net } r_x(M, T) = -p_2 r_2 - p_3 r_3 + \sum_{i=1}^n \text{net } r_{x_i}(\widehat{M}_i, \widehat{T}_i).$$

where  $p_2$  is the number of twice punctured spheres and  $p_3$  is the number of thrice punctured spheres in the decomposition.

*Proof.* The corollary follows immediately from the super-additivity and sub-additivity theorems (Theorems 5.4 and 5.5.) □

It is now easy to verify that net extent and width are additive. Let  $\mathbb{M}$  be the set of irreducible (3-manifold, graph) pairs  $(M, T)$  such that  $T \neq \emptyset$ ,  $T$  intersects every 2-sphere component of  $\partial M$  at least three times, and every  $S^2 \subset M$  separates. Let  $\mathbb{M}_2 \subset \mathbb{M}$  be the subset with elements  $(M, T)$  such that  $g(M) \leq 2$ . Let  $\mathbb{M}_s \subset \mathbb{M}$  be the subset with elements  $(M, T)$  where every closed surface in  $M$  separates.

**Theorem 5.7** (Net Extent and Width are Additive). *Let  $(M, T) \in \mathbb{M}$  be non-trivial, let  $g$  be the Heegaard genus of  $M$ , and let  $x$  be any integer with  $x \geq 2g - 2$ . Then there is a prime factorization of  $(M, T)$  into  $(\widehat{M}_1, \widehat{T}_1), \dots, (\widehat{M}_n, \widehat{T}_n)$  so that there exist integers  $x_1, \dots, x_n$ , summing to at most*



$x - 2(n - 1)$ , with  $x_i$  is realizable for  $(\widehat{M}_i, \widehat{T}_i)$  and

$$\text{netext}_x(M, T) = -p_3/2 + \sum_{i=1}^n \text{netext}_{x_i}(\widehat{M}_i, \widehat{T}_i).$$

where  $p_3$  is the number of thrice punctured spheres in the decomposition. Furthermore, if  $(M, T) \in \mathbb{M}_s$  or if  $(M, T) \in \mathbb{M}_2$  and  $x \leq 2$ , then also

$$w_x(M, T) = -p_3/2 + \sum_{i=1}^n w_{x_i}(\widehat{M}_i, \widehat{T}_i).$$

*Proof.* As in Examples 5.2 and 5.3,  $r = \text{ext}$  and  $\text{net } r = \text{netext}$  satisfy the requirement that for  $S \in \mathbb{S}_0$ ,  $r(S)$  depends only on the euler characteristic and number of punctures of  $S$ . By Corollary 3.3,  $\text{extent}$  is order-preserving. By Corollary 5.6, we have the result for net extent. If  $(M, T) \in \mathbb{M}_s$  or if  $(M, T) \in \mathbb{M}_0$  and  $x \leq 2$ , then a similar argument shows that  $w_x$  is additive.  $\square$

## 6. COMPARISON WITH GABAI THIN POSITION

The width for knots in  $S^3$  defined by Gabai [4] and our definition of  $w_{-2}$  applied to pairs  $(S^3, K)$  are very similar. Both definitions have thick surfaces  $\mathcal{H}^+$  and thin surfaces  $\mathcal{H}^-$  that are spheres and have a height function. Both widths can be calculated via similar formulae. Gabai's width is given [13, Lemma 6.2] by the formula:

$$\frac{1}{2} \left( \sum_{H \sqsubset \mathcal{H}^+} |H \cap K|^2 - \sum_{F \sqsubset \mathcal{H}^-} |F \cap K|^2 \right)$$

and our width is given by

$$2 \left( \sum_{H \sqsubset \mathcal{H}^+} \frac{(|H \cap K| - 2)^2}{4} - \sum_{F \sqsubset \mathcal{H}^-} \frac{(|F \cap K| - 2)^2}{4} \right) = \frac{1}{2} \left( \sum_{H \sqsubset \mathcal{H}^+} (|H \cap K| - 2)^2 - \sum_{F \sqsubset \mathcal{H}^-} (|F \cap K| - 2)^2 \right).$$

Finally, both definitions of width are related to a definition of thin position. Indeed, we can say that  $\mathcal{H}$  is in Gabai thin position if  $\mathcal{H}$  minimizes Gabai's width for a knot  $K$ . Similarly, with our definitions there is always a  $\mathcal{H}$  which is both locally thin and minimizes  $w_{-2}$ .

And yet Gabai thin position is not necessarily additive under connected sum [1] but our width is (Theorem 5.7). The essential difference between the two definitions of width is that in Gabai thin position all the components of  $\mathcal{H} = \mathcal{H}^+ \cup \mathcal{H}^-$  are concentric, while in our definition the components of  $\mathcal{H}$  need not be concentric.

We now briefly examine Blair and Tomova's counterexample to width additivity for Gabai thin position in light of our definition. Figure 8 shows a knot  $K$  (in fact a family of knots) and the connect sum of  $K \# \text{trefoil}$ . Note that the projections of  $K$  and  $K \# \text{trefoil}$  depicted in the figure have the same Gabai width while the trefoil has a Gabai width of 8. The crux of showing that this is indeed a counterexample to additivity of Gabai width is to show that the embedding of  $K$  depicted in Figure 8 is actually in Gabai thin position. The thin and thick surfaces in the figure are a v.p.-multiple bridge surface  $\mathcal{H}$ . As  $w_{-2}$  is additive, it must be the case that  $\mathcal{H}$  is not a minimum width multiple v.p. bridge surface for  $K$ . Note that

$$w(\mathcal{H}) = 2(4^2 + 4^2 + 4^2 - 1^2 - 1^2) = 92$$

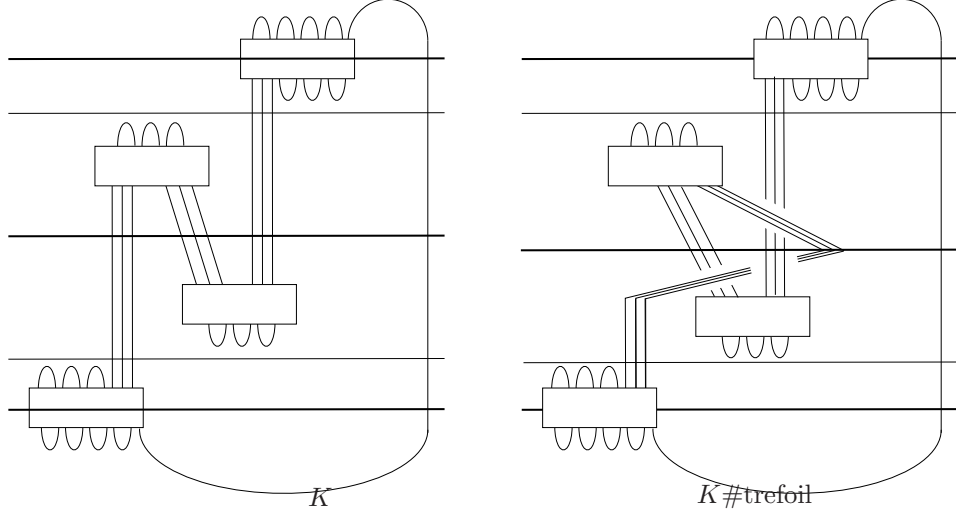


FIGURE 8. The rectangles represent particular braids, which are irrelevant for our purposes. Thick and thin surfaces are represented with thick and thin lines respectively.

Another projection  $K'$  of the knot  $K$  is depicted on both the left and right of Figure 9. That  $K$  and  $K'$  are isotopic was noted by Scharlemann and Thompson in [16]. To show that the multiple bridge surface  $\mathcal{H}$  on the left of Figure 9 is not locally thin, we point out (again on the left of Figure 9) a weak-reducing pair of discs for each thick surface. Applying two elementary thinning sequences using the indicated discs produces (after an isotopy) the multiple v.p. bridge surface  $\mathcal{H}'$  depicted on the right of Figure 9. Using our formula for width

$$w(\mathcal{H}') = 2(2^2 + 4^2 + 4^2 + 2^2 - 1^2 - 1^2 - 1^2) = 74.$$

This demonstrates that  $\mathcal{H}$  is indeed not a minimum width multiple v.p. bridge surface for  $K$ , although it does minimize Gabai width (for particular choices of braids.)

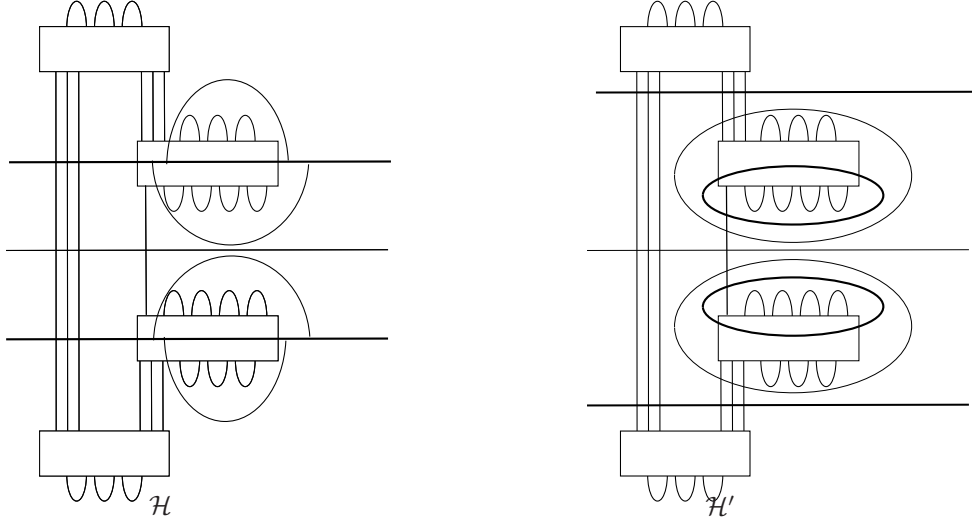


FIGURE 9. Thick and thin surfaces are represented with thick and thin lines respectively.

## 7. ON SOME CLASSICAL INVARIANTS

As an easy example of how net extent can be used to study classical invariants, we reprove classical theorems of Schubert [17] and Norwood [11]. The Schubert theorem is also a consequence of the fact that the double branched cover over a 2-bridge knot is a lens space. We include a proof, however, as an example of how to use our techniques.

**Theorem 7.1** (Schubert). *If  $K \subset S^3$  is a knot which is 2-bridge with respect to a sphere, then  $K$  is prime.*

*Proof.* Suppose that  $K$  is a composite 2-bridge knot. Since  $K$  is 2-bridge,

$$\text{ext}_{-2}(S^3, K) \leq 1.$$

Since extent is always integral for knots in  $S^3$ , by Theorem 4.10,  $\text{ext}_{-2}(S^3, K) = 1$ . Since  $K$  is composite, by Theorem 5.7, it has a prime factorization  $K = K_1 \# \dots \# K_n$  such that

$$1 = \text{netext}_{-2}(S^3, K) = \text{netext}_{-2}(S^3, K_1) + \dots + \text{netext}_{-2}(S^3, K_n).$$

Since each  $K_i$  is non-trivial, by Theorem 4.10 it follows that:

$$1 \geq n.$$

Thus,  $n = 1$  and so  $K$  is prime. □

Recall that a Heegaard surface for the exterior of a knot  $K$  in a 3-manifold  $M$  is a linear, oriented, multiple v.p.-bridge surface for  $(M, K)$ . Thus,  $\text{netext}_\infty(M, K) \leq t(K)$ .

**Theorem 7.2** (Norwood). *If  $K \subset S^3$  has  $t(K) = 1$ , then  $K$  is prime.*

*Proof.* Let  $K$  be a knot with  $t(K) = 1$ . Since the unknot has tunnel number 0,  $K$  is not the unknot. By Theorem 5.7,  $K$  has a prime factorization

$$K = K_1 \# \dots \# K_n$$

such that

$$1 \geq \text{netext}_x(K) = \text{netext}_x(K_1) + \dots + \text{netext}_x(K_n).$$

Since each  $K_i$  is non-trivial, by Theorem 4.10 it follows that:

$$1 \geq n.$$

Thus,  $n = 1$  and so  $K$  is prime. □

Scharlemann and Schultens [14] generalized Norwood's theorem to show that if a knot  $K \subset S^3$  has at least  $n$  prime factors, then  $K$  has tunnel number at least  $n$ . (Another proof has been given by Weidmann [21].) Morimoto [10] showed that the tunnel number of meridionally small knots does not go down under connected sum. Here is a common generalization of both the Scharlemann-Schultens and Morimoto result.

**Theorem 7.3.** *For each  $i \in \{1, \dots, n\}$  let  $K_i$  be a knot in a closed, orientable 3-manifold  $M_i$  such that every sphere in  $M_i$  separates and each  $(M_i, K_i)$  is prime. Assume that there is an integer  $j \leq n$  is such that  $(M_i, K_i)$  is meridionally small if and only if  $i \leq j$ . Then, letting  $(M, K) = (M_1, K_1) \# \dots \# (M_n, K_n)$ , we have:*

$$(n - j) + t(K_1) + \dots + t(K_j) \leq t(K) \leq (n - 1) + \sum t(K_i)$$

*Proof.* Since every sphere in every  $M_i$  separates, each pair  $(M_i, K_i)$  is irreducible. Furthermore, by [9, Theorem 4.1], the prime factorization

$$(M_1, K_1) \# \cdots \# (M_n, K_n)$$

is unique up to re-ordering.

Let  $t$  be the tunnel number of  $K$  and let  $H$  be a minimal genus Heegaard surface for the exterior of  $K$ . Let  $x = \infty$ . The surface  $H$  is also a v.p.-bridge surface for  $(S^3, K)$  and so

$$(3) \quad \text{netext}_x(M, K) \leq \text{ext}(H) = t(K).$$

By Theorem 5.7 and our assumption on the uniqueness of prime factorization, there exist  $(M_1, K_1), \dots, (M_n, K_n)$  such that there are integers  $x_1, \dots, x_n$  so that all of the following hold:

- (1)  $x_i$  is realizable for  $(M_i, K_i)$  for all  $i$ .
- (2)  $x_1 + \dots + x_n \leq x - 2(n - 1)$
- (3)

$$\text{netext}_{-2}(M, K) = \sum_{i=1}^n \text{netext}_{x_i}(M_i, K_i).$$

Suppose  $(M_i, K_i)$  is meridionally small and let  $\mathcal{H}_i$  be a multiple v.p.-bridge surface for  $(M_i, K_i)$  such that  $\text{net}\chi(\mathcal{H}_i) \leq x_i$  and  $\text{netext}(\mathcal{H}_i) = \text{netext}_{x_i}(M_i, K_i)$ . If there were a component of  $\mathcal{H}_i^-$  which intersected  $K_i$ , by Theorem 2.11, we would contradict local thinness. Thus,  $K \cap \mathcal{H}_i^- = \emptyset$  and there is at most one component  $H \subset \mathcal{H}_i^+$  which intersects  $H$ . By performing  $|H \cap K_i|/2$  meridional stabilizations on  $H$ , we may create a surface  $H'$  such that  $\mathcal{H}'_i = (\mathcal{H}_i \setminus H) \cup H'$  is a multiple v.p.-bridge surface for  $(M_i, K_i)$  disjoint from  $K_i$ . Observe that  $\text{netext}(\mathcal{H}'_i) = \text{netext}(\mathcal{H}_i)$ . Since  $\mathcal{H}'_i$  is disjoint from  $K_i$  we may amalgamate [18]  $\mathcal{H}'_i$  to a Heegaard surface  $J_i$  for the exterior of  $K_i$ . It is easy to verify that

$$(4) \quad t(K_i) \leq -\chi(J_i)/2 = \text{ext}(J_i) = \text{netext}(\mathcal{H}'_i) = \text{netext}_{x_i}(M_i, K_i).$$

Observe that if  $(M_i, K_i)$  is (lens space, core loop), then  $t(K_i) = 0 = \text{netext}_{x_i}(M_i, K_i)$ .

By Theorem 4.10, we have  $\text{netext}_{x_i}(M_i, K_i) > 0$  whenever  $(M_i, K_i)$  is not  $(S^3, \text{unknot})$  or (lens space, core loop). Combining Equation (3), Equation (3), and inequality we obtain:

$$t(K) \geq t(K_1) + \dots + t(K_n) + m.$$

Finally, a standard construction shows that  $t(K) \leq t(K_1) + \dots + t(K_n) + (n - 1)$ , completing the proof.  $\square$

For our final application, we show that higher genus bridge number, together with the genus, is super-additive under connected sum of knots that are small and m-small. A more detailed analysis would likely produce an even stronger result.

**Theorem 7.4.** *Suppose that  $(\widehat{M}_i, \widehat{K}_i)$  are small and m-small for  $i \in \{1, \dots, n\}$ . Let  $(M, K) = \#_{i=1}^n (\widehat{M}_i, \widehat{K}_i)$  and let  $g \geq g(M)$ . Then there exists  $g_i$  such that  $\sum g_i \leq g$ ,  $g(\widehat{M}_i) \leq g_i$  and*

$$\sum_{i=1}^n (g_i + b_{g_i}(K) - 1) \leq g + b_g(K) - 1.$$

In the following we again implicitly use the uniqueness of prime factorization.

*Proof.* Let  $S$  be a genus  $g$  bridge surface for  $(M, K)$  realizing  $b_g(K)$ . We may perform a sequence of generalized destabilizations, undoing of removable edges, and unperturbations to arrive at a reduced v.p.-bridge surface  $H$  for  $(M, K)$  with  $g(H) \leq g$ . Let  $\mathcal{H} \in \mathbb{H}(M, K)$  be a locally thin multiple v.p.-bridge surface for  $(M, K)$  such that  $H \rightarrow \mathcal{H}$ . Recall that  $\text{net } \chi(\mathcal{H}) \leq \text{net } \chi(H)$ . Let  $\mathcal{Q}$  be the union of twice and thrice-punctured spheres in  $\mathcal{H}^-$ . By Theorem 5.1 some subset of  $\mathcal{Q}$  is the union of summing spheres giving a prime decomposition of  $(M, K)$ . Let  $(M', T') = (M, T) \setminus \mathcal{Q}$  and let  $(\widehat{M}, \widehat{T})$  be the result of capping off the components of  $\partial M$  corresponding to  $\mathcal{Q}$ . Then  $(\widehat{M}, \widehat{T})$  is the union of summands  $(\widehat{M}_i, \widehat{K}_i)$  for  $i \in \{1, \dots, n\}$  and the union of  $(S^3, \text{unknot})$  pairs.

Suppose that  $F \sqsubset \mathcal{H}^- \setminus \mathcal{Q}$  is contained in the interior of some  $(M_i, K_i)$ . By Theorem 2.11,  $F$  is essential in  $(M, K)$ . If  $F$  is not essential in  $(\widehat{M}_i, \widehat{K}_i)$  then it must be  $\partial$ -parallel in  $\widehat{M}_i \setminus \eta(\widehat{K}_i)$ . However, since  $\widehat{K}_i$  is a knot, this implies that  $F$  is a sphere intersecting  $K$  twice. By the definition of  $\mathcal{Q}$ , this implies  $F \sqsubset \mathcal{Q}$ , a contradiction since  $F$  is in the interior of  $M_i$ . Thus,  $F$  is essential in  $(\widehat{M}_i, \widehat{K}_i)$ . Since  $(\widehat{M}_i, \widehat{K}_i)$  is small and m-small, the surface  $F$  cannot exist, and so  $\mathcal{H}^-$  is disjoint from the interior of each  $(M_i, K_i)$ .

We conclude, therefore, that each  $(\widehat{M}_i, \widehat{K}_i)$  contains exactly one component  $H_i$  of  $\mathcal{H}^+$ , and  $H_i$  is a v.p.-bridge surface for  $\widehat{M}_i$ . Let  $g_i = g(H_i)$ . Observe that  $g_1 + \dots + g_n \leq g$ . Since  $\text{netext}_\infty$  is non-negative for each component of  $(\widehat{M}, \widehat{K})$ , we have:

$$\sum_{i=1}^n g_i + |H_i \cap K_i|/2 - 1 = \sum_{i=1}^n \text{ext}(H_i) \leq \text{netext}(\mathcal{H}) \leq \text{ext}(S) = g + b_g(K) - 1.$$

Thus,

$$\sum_{i=1}^n (g_i + b_{g_i}(K) - 1) \leq g + b_g(K) - 1.$$

as desired. □

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